ANALYSIS OF TWO-WAY SLAB USING YIELD LINE METHOD

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مُسْـتَخْلَـص

تعنى هذه الورقة بتحليل خط الخضوع للبلاطة اتجاهين من الخرسانة المسلحة تقع تحت تأثير الحمل الموزع بانتظام على مساحة البلاطة. اعتمد التحليل على الطريقة التي تبناها يوهانسون، والتي استنتجت فيها صيغة عامة لحساب أقصى عزم انحناء موجب بالبحر الطويل. تعتمد الصيغة العامة بشكل أساسي على الأبعاد الهندسية الناتجة عن نمط الانهيار لخطوط الخضوع بالإضافة إلى معاملات العزوم التي استخدمت للربط بين العزم المحسوب والعزوم الأخرى. تم اشتقاق معاملات العزوم رقميًا باستخدام برنامج STAAD-Pro من خلال تبني تسع حالات من الحالات الحدودية باستخدام نسب أطوال بحور مختلفة تراوحت من 1.0 إلى 2.0. بالنسبة للحالات التسعة ومع استخدام نسب البحور المختلفة، تم حساب عزوم الانحناء الحدية باستخدام طريقة خط الخضوع. وتمت مقارنة النتائج التي تم الحصول عليها مع نتائج استخرجت من المدونة BS8110 بالإضافة إلى تلك التي تم الحصول عليها باستخدام برنامج STAAD-Pro.

ABSTRACT

This paper deals with the yield line analysis of orthotropic reinforced concrete two-way slab under the effect of uniformly distributed pressure load. The analysis was based on the method developed by Johansson, in which a general formula was derived to calculate the ultimate positive bending moment for the long span. The general formula depends mainly on the geometric dimensions resulting from the yield lines pattern as well as the moment's coefficients that have been used to relate the calculated moments with other moments. The moment's coefficients have been derived numerically using STAAD-Pro Software by adopting nine cases of boundary conditions with using different spans ratios range from 1.0 to 2.0. For the nine cases and with using different spans ratios, the ultimate bending moments have been calculated using yield line method. The results obtained were compared to that extracted from the BS8110 Code as well as those obtained using STAAD-Pro Software.

Keywords *Yield line method, Two-way Slab, Analysis of slab, Slab moment's coefficients.*

1 Introduction

The reinforced concrete slabs are important structural members because they carry the transverse loads of the buildings directly and in turn, it resists these loads by bending action either in one direction or in two directions. Slabs are classified according to the supporting conditions and according to their composition to many different types. The analysis of slabs is somewhat complex and there are many methods used in the analysis of slabs, including analytical methods and numerical methods, based on the properties of concrete in terms of elasticity and/or plasticity. The ACI, BS8110 and European Standards, established coefficients for calculating bending moments and shear forces for various slab cases according to the supporting conditions. But using these coefficients is subject to conditions that must be met prior to use. One method that has recently been used and found acceptable in the British, European and American Standards is the Yield Line Method, which is the method classified as ultimate limit state method. The yield line is economically advantageous because the moment calculated by it is less than the calculated by any other methods.

Yield-line analysis for slabs was initiated by Ingerslev (1923) and was extended greatly by Johansen (1943, 1949) [1]. Its main application to reinforced concrete slabs whose structural characteristics are dominated by yielding of the steel reinforcement [2]. The guidance document produced by the U.K. Concrete Centre (Kennedy and Goodchild 2004) discusses the many benefits of yield-line design, in particular highlighting the highly economic reinforcement layouts that can result from its application [3] though it should be noted that the method considers flexural failure only, and serviceability considerations, which will sometimes govern the design, are not considered [4]. Due to the upper-bound behaviors of the yield-line method, a yield-line patterns will often need to be explored, which can be time-consuming. Furthermore, there is often the concern that the critical pattern may have been missed, and consequently that an unsafe load carrying capacity has been computed [5]. The basic assumption of the yield-line theory, first developed by Johansen, is that a reinforced concrete slab, similar to a continuous beam or frame of a perfectly plastic material, will develop yield hinges under overload, but will not collapse until a mechanism is formed. [6,7].

To get yield line solution, there may be several possible valid yield line patterns that could apply to a particular configuration of a slab and loading. However, there is one yield line pattern that gives the highest moments or least load at failure [3]. The solution can be carried out by the equilibrium method, in which equilibrium equations are written for each plate segment, or by the virtual-work method, in which some part of the slab is given a virtual displacement and the resulting work is considered.

In yield line the slab can be described as isotropic slab if the same amount of bottom reinforcement both ways, or orthotropic slabs which have different amounts of reinforcement in the two directions [3].

A 10% margin on the ultimate moments should be added to two-way slabs to allow for the effects of corner levers [3,8].

STAAD Pro is a general-purpose program for performing the analysis and design of a wide variety of types of structures. The modeling and analysis of a slab and other surface entities like walls are modeled using plate elements which are using generation method for generating the finite element model.

In this work, an analysis of two-way reinforced concrete slab has been done and the solution was carried out using virtual work method by adopting general case of slab probable different cases of slabs according to the supporting conditions as reported by the BS8110.

1.1 Aims and Objective

This paper aims to analyze two-way reinforced concrete slab using yield line theory in order to:

1. Express a general formula for bending moment through following the procedure of yield line method.
2. Deduce the moment's coefficients through studying the relation between the two-way slab bending moments using STAAD-Pro Software.
3. Calculate the ultimate bending moments for the two-way slabs using the general formula and compared the results obtained with those obtained using STAAD-Pro and BS8110 Code.

2.0 The Yield Line Theory

At failure, the yield lines divide the slab into several segments and all rotations take place in yield lines. By choosing some convenient point as point of maximum deflection δ and normally is assumed as unit value and according to the principal of virtual load, external work done by applied loads is equated to the internal work done along yield-lines as shown in Equation 1.

$$\sum w\,\delta = \sum m\,l\,\theta \qquad (1)$$

Where:

w is the Load acting within a particular segment

δ is the vertical displacement of the load w on each segment expressed as a fraction of unity

m is the moment or moment of resistance of the slab per meter run represented by the reinforcement crossing the yield line *l is the length of yield line or its projected length onto the axis of rotation for that segment*

θ is the rotation of the segment about its axis of rotation

The moment across the yield lines being a maximum value, the correct yield pattern, corresponding to a load *w* will give a maximum value of *m* from Equation l as compared to other patterns. If a type of pattern is assumed in accord with the support conditions and characterized by a number of unknown parameters $x_1, x_2, \ldots x_n$ Equation l can be written by:

$$m = f(x_1.x_2.\ldots.x_n) \qquad (2)$$

The correct yield pattern then is formed by the maximum criteria:

$$\frac{\partial f}{\partial x_1} = 0. \quad \frac{\partial f}{\partial x_2} = 0. \ldots. \quad \frac{\partial f}{\partial x_n} = 0 \qquad (3)$$

The final yield moment *m* is determined by substituting the corresponding parameter values into Equation 2.

3.0 General Cases for Uniformly Loaded Two-Way Slabs

The general cases of uniformly loaded two-way slabs will be considered. The slabs will be considered to be orthotropically reinforced. The slab and the yield line pattern are shown in Figure 1. All edges of slab are assumed to be fixed and the ultimate negative moments and ultimate positive moment for short span are defined in terms of positive moment for long span.

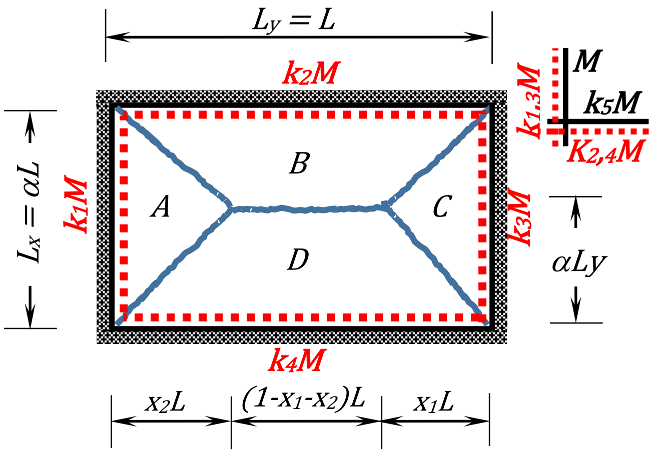


Figure 1: Yield Line Pattern for the General Case of Two-Way Slab

Where:

α is the ratio between 0.5 – 1.0 used to calculate short span as a ratio of long span.

L and αL are the dimensions of slab, long span (L_y) and short span (L_x) respectively.

x_1L, x_2L, and αLy are unknown dimensions define the location of yield lines.

k_1, k_2, k_3, and k_4 are the fixity ratios for the four edges also can be defined as the negative moment coefficients.

k_5 is the positive moment coefficient for short span

M is the ultimate positive bending moment per unit length for the long span.

A, B, C, and D are the slab segments due to yield line pattern.

——— *is the axis of rotation for the positive moment.*

······· *is the axis of rotation for the negative moment.*

In order to generate the nine cases as stated by BS8110 code, the four edges can be altered between fixed and simply supported. The case of a simply supported edge can be obtained by putting the fixity ratio equal to zero. A fixed edge means continuous edge with a negative moment. And a simply supported edge means discontinuous edge with zero negative moment.

3.1 Moment's Coefficients

According to Reference [3] and [6], the moment's coefficients are assumed to be chosen by the designer firstly. In this paper, the fixity coefficients at edges as well as the moment's coefficient for the short span are estimated by studying results obtained using finite element method through using STAAD-Pro software. Nine cases were adopted attempting different edges conditions, as well as different ratios between the two spans of the slab using parameter (α), ranged between 0.5 and 1.0. The positive moment for the long span is the lowest moment among others, so is taken as the base for obtaining the moment's coefficients which are calculated according to Equation (4).

$$k_1 = k_3 = \frac{M_y(Neg)}{M_y(Pos)} \; ; \; k_2 = k_4 = \frac{M_x(Neg)}{M_y(Pos)};$$

$$k_5 = \frac{M_x(Pos)}{M_y(Pos)} \qquad (4)$$

Where:

k_1 to k_5 as shown in Figure (1).

M_x is the moment for the short span.

M_y *is the moment for the long span.*

In order to express the values of the moment's coefficients in an easy and practical way, a link was obtained between them and spans ratios using a specialized program CurveExpert, and the best model that has been found to relate them is a quadratic formula as shown in Equation (5).

$$\left.\begin{aligned} k_1 = k_3 &= 0.21 + 1.68R - 0.48R^2 \\ k_2 = k_4 &= -2.74 + 5.41R - 1.09R^2 \\ k_5 &= -1.53 + 3.0R - 0.47R^2 \end{aligned}\right\} \quad (5)$$

Where R is span ratio for slab

$$R = \frac{L_y}{L_x} = \frac{L}{\alpha L} = \frac{1}{\alpha}$$

3.2 General Formula for Bending Moment

According to the yield line pattern shown in Figure 1, the bending moment can be derived by applying the concepts of virtual work and substituting in Equation 1. The internal and external work done can be obtained by follow the same procedure stated in most of the References listed, at final the following expressions were obtained.

$$Total\ Internal\ Work\ Done = \frac{\alpha M(1+k_1)}{x_2} + \frac{\alpha M(1+k_3)}{x_1} + \frac{M(k_2+k_5)}{\alpha(1-y)} + \frac{M(k_4+k_5)}{\alpha y} \quad (6)$$

$$Total\ External\ Work\ Done = ⅙\alpha wL^2(3 - x_1 - x_2) \quad (7)$$

By applying Equations (6) and (7) in Equation (1), the bending moment is found as shown in Equation (8).

$$M = \frac{w(\alpha L)^2}{6}\left(\frac{(3 - x_1 - x_2)}{t_1 + t_2 + t_3 + t_4}\right) \quad (8)$$

where:

$$t_1 = \frac{\alpha^2(1+k_1)}{x_2} \ ; \ t_2 = \frac{\alpha^2(1+k_3)}{x_1}$$

$$t_3 = \frac{(k_2+k_5)}{(1-y)} \ ; \ t_4 = \frac{(k_4+k_5)}{y}$$

The ultimate bending moment can be calculated according to the values of parameters x_1, x_2 and y which have been estimated using the concept explained in Equation (3). y is given by Equation (9) and totally is dependent on moment's coefficients. x_1 and x_2 are calculated simultaneously using Equation (10) and using excessive calculation aided by spreadsheets in order to give ultimate value of *M*.

$$y = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad (9)$$

Where:

$$a = k_4 - k_2 \ ; \ b = -(2k_4 + 2k_5) \ ;$$

$$c = (k_4 + k_5)$$

$$x_1 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad (10)$$

Where:

$$a = \big(-s(1+k_1) - x_2 y(k_2+k_5) - x_2(1-y)(k_4+k_5)\big)$$

$$b = -2\big(sx_2(1+k_3)\big) \ ; \ c = sx_2(1+k_3)(3-x_2)$$

4.0 Calculation of the Ultimate Bending Moments for the Different Cases of the Slab

The cases taken here were the nine cases listed in BS8110, these cases are shown in Table 1. For the different values of the span's ratios (R) which ranged between 1.0 and 2.0, the moment's coefficients have been determined covering the nine cases of the slab using Equation (5). Again, the Curvexpert program was used to relate the yield line dimensions x_1, x_2 *and y* with R and the Equations obtained were listed in Table 2. The positive bending moments for the long span can be calculated using Equations (8), (9) and (10) and for simplification, quadratic equations dependent on R have been derived and listed in Table 2. The values of bending moments obtained were used to calculate the other bending moments using Equation (11).

$$M_x(Pos) = k_5 M_y(Pos);$$

$$M_x(Neg) = k_{2.4} M_y(Pos) \ ;$$

$$M_y(Neg) = k_{1.3} M_y(Pos) \quad (11)$$

Table 1: The Nine Cases of Slab According to BS8110

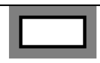
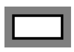
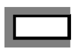
Case No.	Description	Figure
1	Interior panels (4-Edges Continues)	
2	One short edge discontinuous	
3	One long edge discontinuous	
4	Two adjacent edges discontinuous (Corner)	
5	Two short edges discontinuous	
6	Two long edges discontinuous	
7	Three edges discontinuous (one long edge continuous)	
8	Three edges discontinuous (one short edge continuous)	
9	Four edges discontinuous	

Table 2: Equations for Calculating Yield Line Dimensions and the Ultimate Positive Moment for Long Span for the Nine Cases

Case	*y*	***x₁***	***x₂***	**M**
	0.5	$x_1 = 1.050 - 0.717R + 0.153R^2$	$x_2 = x_1$	$M = 0.024 - 0.007R + 0.001R^2$
	0.5	$x_1 = 0.874 - 0.692R + 0.165R^2$	$x_2 = 1.230 - 0.900R + 0.202R^2$	$M = 0.035 - 0.017R + 0.004R^2$
	0.62	$x_1 = 1.070 - 0.663R + 0.133R^2$	$x_2 = x_1$	$M = 0.020 + 0.003R - 0.001R^2$
	0.62	$x_1 = 0.852 - 0.612R + 0.136R^2$	$x_2 = 1.260 - 0.840R + 0.175R^2$	$M = 0.036 - 0.009R + 0.001R^2$
	0.5	$x_1 = 0.954 - 0.760R + 0.179R^2$	$x_2 = x_1$	$M = 0.048 - 0.028R + 0.006R^2$
	0.5	$x_1 = 1.080 - 0.581R + 0.103R^2$	$x_2 = x_1$	$M = 0.010 - 0.024R + 0.007R^2$
	0.62	$x_1 = 1.05 - 0.813R + 0.190R^2$	$x_2 = x_1$	$M = 0.053 - 0.023R + 0.004R^2$
	0.5	$x_1 = 0.884 - 0.572R + 0.118R^2$	$x_2 = 1.31 - 0.776R + 0.149R^2$	$M = 0.041 - 0.004R$
	0.5	$x_1 = 1.15 - 0.84R + 0.19R^2$	$x_2 = x_1$	$M = 0.053 - 0.007R$

5.0 Verification of the Ultimate Bending Moments produced by Yield Line Method

The ultimate bending moments obtained by using yield line theory were compared with those obtained by using STAAD-Pro Software and with those extracted from the BS8110. The comparison has been done using graphs include all nine cases and samples of these graphs were illustrated as shown in Figures (2-7).

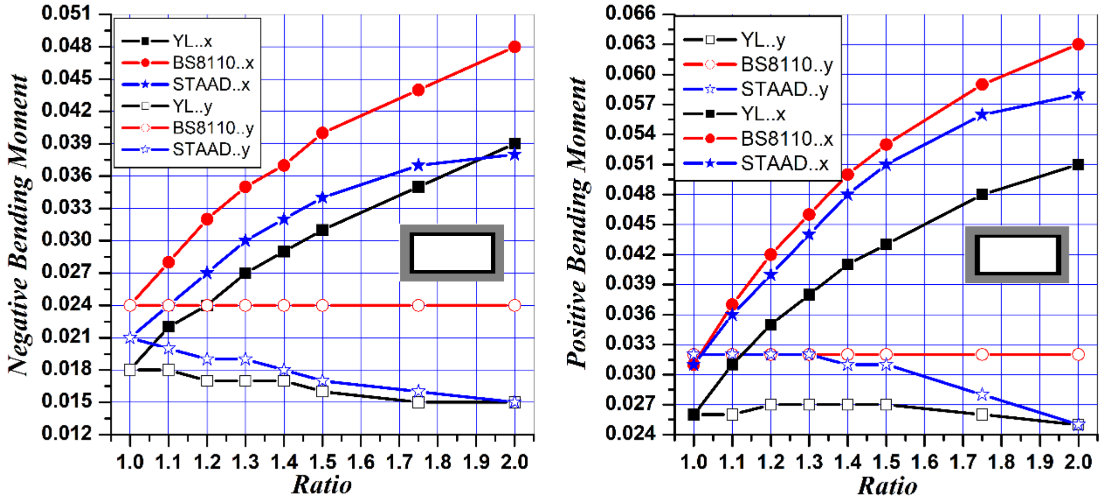


Figure 2: Bending Moment for Interior Slab

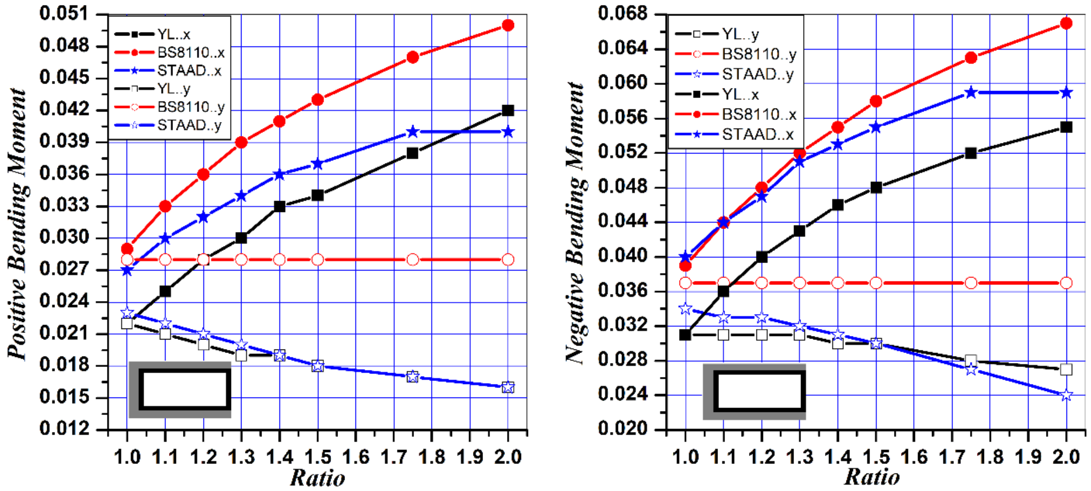


Figure 3: Bending Moment for One Short Edge Discontinuous Slab

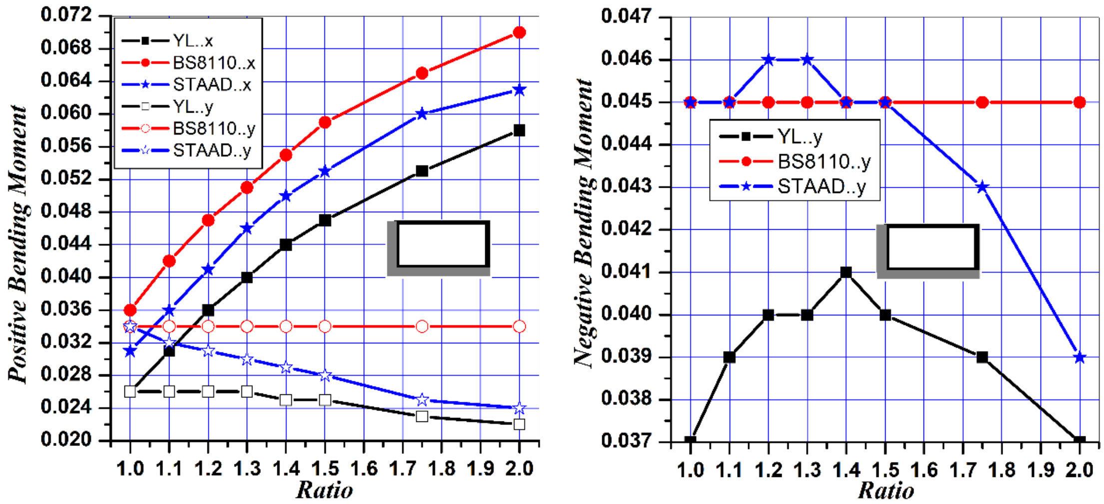


Figure 4: Bending Moment for Two Adjacent Edges Slab

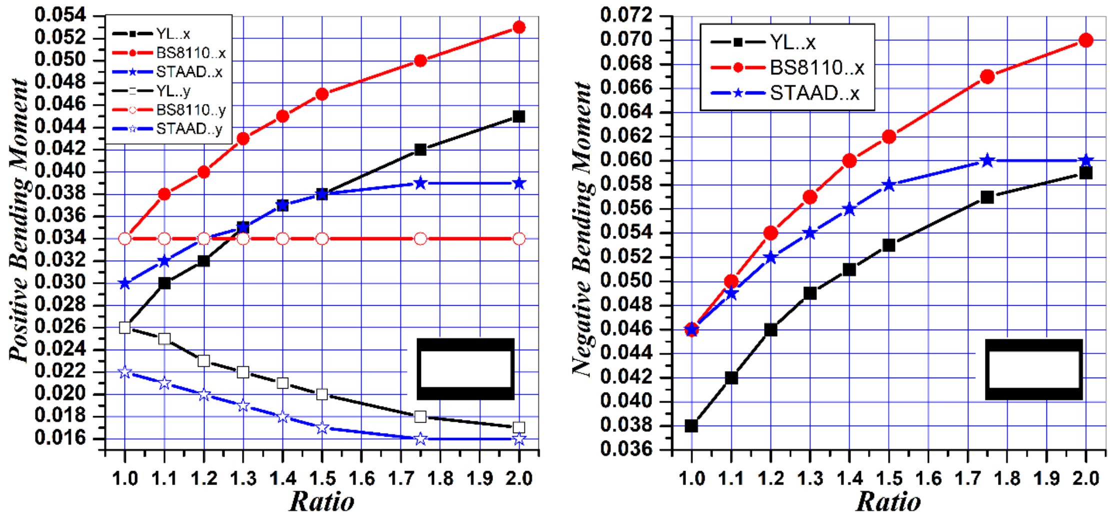


Figure 5: Bending Moment for Two Short Edges Discontinuous Slab

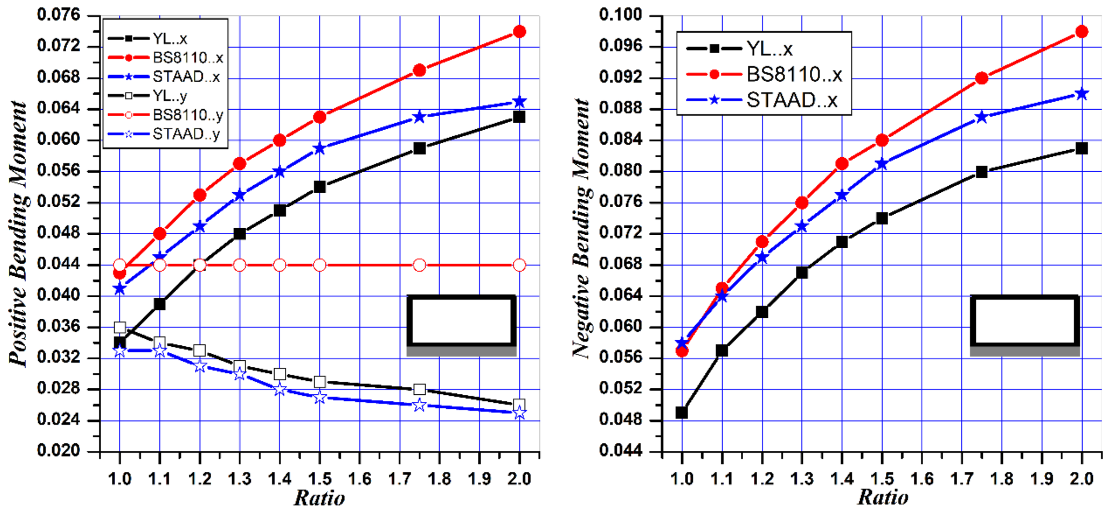


Figure 6: Bending Moment for One Long Edge Continuous Slab

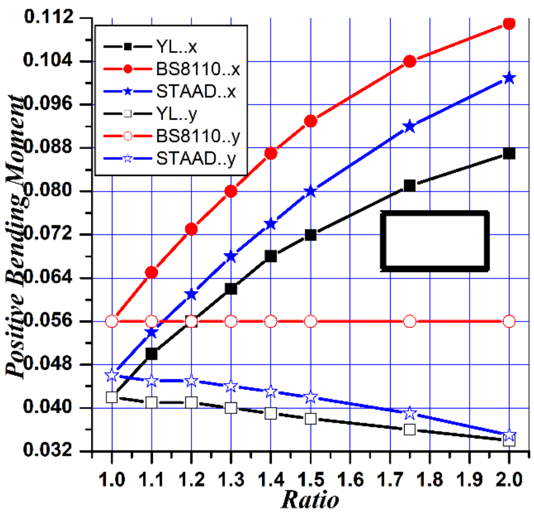


Figure 7: Bending Moment for Four Edge Discontinuous Slab

For the Figures 2-7, the notations used in the legends are defined as follow.

YL.x, YL.y are referred to the moment calculated according to the yield line theory along short span and long span respectively.

BS8110.x, BS8110.y are referred to the moment extracted from BS8110 Code along short span and long span respectively.

STAAD.x, STAAD.y are referred to the moment Calculated using STAAD Pro Software along short span and long span respectively

6.0 Results and Discussion

As clear from the above figures, the bending moments obtained using yield line theory, it is found always less than those obtained by BS8110 and STAAD-Pro by an amount range between 15% to 30%. This is consistent with the literature reviewed which emphasized that the moment obtained using yield line is more economical than that obtained by any other method. The bending moment for short span, it always increases with the span ratio increased, while for the long span it found decreases, this is in line with the well-known concept of the two-way slab. It is optional to calculate the bending moments; either using the simplified equations listed in Table 2 or extracted it directly from the figures.

7.0 Conclusion

The yield line theory has been conducted in this paper for the two-way reinforced concrete slab. A general case of the slab has been analyzed and the calculations were carried out to estimate the values of yield line dimensions firstly and then the ultimate bending moments can be calculated. The bending moment for the long span is always is the less one. Because of this reason, all the slab moments were taken as a ratio to this moment. The calculated dimensions, as well as the bending moment, have been articulated with the span's ratios, by quadratic equations which lead to simple calculations.

According to the results obtained, we concluded that:

- The Bending moment calculated using yield line theory is more economical than the other methods.
- For the short span, the average percentage difference between the ultimate positive moment obtained by using yield line and by using BS8110 is about 22% less.
- For the long span, the average percentage difference between the ultimate positive moment obtained by using yield line and by using BS8110 is about 30% less.
- For the short span and long span, the average percentage difference between the ultimate negative moment obtained by using yield line and by using BS8110 is about 15% less.

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