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Optimum Basis for Turbulent Shear Flow

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Abstract: Proper Orthogonal Decomposition (POD) eigenfunctions have many useful applications in the analysis of turbulent shear flows, e.g., reconstruction of flow statistics, investigation of coherent structures, basis for low dimensional modelling (LDM), and flow control. The major shortcoming of POD method and its implementation to LDM and flow control is the dependence of POD basis on the flow parameters and geometry from which they were extracted. The paper develops a proposal for a model of energetic-structures in which the cross-spectral tensor is determined from the solution of spectral equations. The obtained energetic-structures are similar to those observed in experiments or extracted from numerical simulations data.

Keywords: *Coherent structures; Flow control; Dynamical systems; Turbulence modelling.*

1. INTRODUCTION

Because of its simple geometry, fully developed channel flow Fig. 1 has been investigated extensively both numerically and experimentally. The quantitative analysis of turbulent flows utilizing proper orthogonal decomposition has been an active area of research in the last 20 years. Most of works have concentrated on the analysis of time series data from laboratory experiments or numerical simulations. The main objectives are of two folds; reconstruction of coherent structures and extraction of optimal basis for low dimensional modelling (LDM). Early application of POD to wall bounded flows was reported by Herzog [1], he extracted an optimal POD basis which was later used by Aubry *et al.* [2] to construct a 10-D low dimensional model for a sub domain in the wall-normal direction $0 \le y^+ \le 60$. Herzog's work on POD and Aubry's work on LDM were continued by Moin and Moser [3], Berkooz *et al.* [4], Poje and Lumley [5], Prabhu *et al.* [6], and Smith *et al.* [7].

The major shortcoming of POD method and its implementation to LDM is the dependence of POD basis on the flow parameters and geometry from which they were extracted. However, successful works in kinetic energy analysis, extraction of coherent structures, and low dimensional description of near-wall turbulence are reported continuously. The reader is referred to Lumley [8-10], Sirovich [11], and Holmes *et al.* [12] for more details.

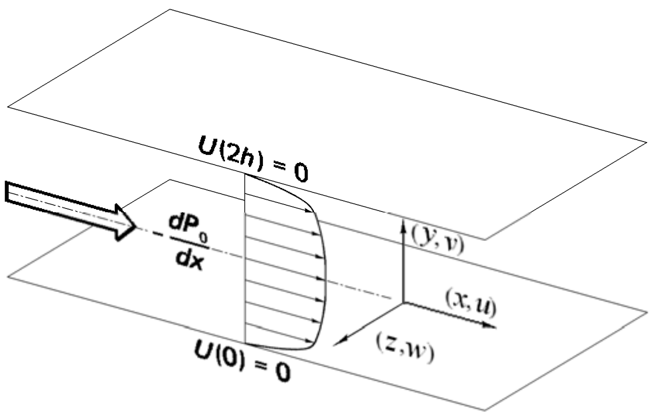

Fig. 1. Sketch of the flow geometry

2. PROPER ORTHOGONAL DECOMPOSITION

Suppose we have a random velocity field, $u_i(\cdot)$. We seek to find a deterministic vector field $\phi_i(\cdot)$ which has the maximum projection on our random vector field u_i; in a mean square sense. We would like to find a whole new deterministic field represented by $\phi_i(\cdot)$ for which $<|\alpha|^2> = <|u_i(\cdot)\phi_i^*(\cdot)|^2>$ is maximized, i.e.

$$<|\gamma|^2> = \frac{\langle(\phi_i(\cdot), u_i(\cdot))^2\rangle}{(\phi_i(\cdot), \phi_i(\cdot))} \tag{1}$$

$$\int\int_D R_{ij}(\cdot,\cdot')\phi_i^*(\cdot)\phi_j(\cdot')d(\cdot)d(\cdot') = \lambda\int_D \phi_i(\cdot)\phi_i^*(\cdot)d(\cdot) \qquad (2)$$

where $\lambda =< |\alpha|^2 >$. So, if $\phi_i(\cdot)$ maximize (2), it means that if the flow field is "projected" along $\phi_i(\cdot)$, the average energy content (λ) is larger than if the flow field is "projected" along any other mathematical structure, e.g. a Fourier mode. In the space orthogonal to this $\phi_i(\cdot)$ the maximization process can be repeated, and in this way a whole set of orthogonal functions $\phi_i(\cdot)$ can be determined. This method is called proper orthogonal decomposition, or POD. The power of POD lies in the fact that the decomposition of the flow field in the POD eigenfunctions converge optimally fast in L^2-sense. Most importantly, the decomposition is based on the flow field itself,. If the flow field is inhomogeneous of finite extent, then Hilbert-Scmidt theory applies and the obtained eigenfunctions are empirical, while if the flow field is homogenous or periodic of infinite extent the eigenfunctions are analytical (sines and cosines).

A necessary condition for $\phi_i(\cdot)$ to maximize expression (2) is that it is a solution of the following Fredholm integral equation of the second type

$$\int_D R_{ij}(\cdot,\cdot')\phi_j(\cdot')d(\cdot') = \lambda\phi_i(\cdot) \qquad (3)$$

where, R_{ij} is the space-correlation tensor. We can use the eigenfunction as a basis for the flow field.

$$u_i(\cdot) = \sum_{n=0}^{\infty} a_n\phi_i^n(\cdot) \qquad (4)$$

The random coefficients a_n are determined by projection back onto the velocity field i.e.,

$$a_n = \int_D u_i(\cdot)\phi_j^{(n)*}(\cdot)d(\cdot) \qquad (5)$$

They are uncorrelated and their mean values are the eigenvalues λ

$$\lambda_n =< a_n a_m > \delta_{nm} \qquad (6)$$

The eigenvalues are ordered (meaning that the lowest order eigenvalue is bigger than the next, and so on); i.e, $\lambda_1 > \lambda_2 > \lambda_3 \cdots$. Thus the representation is optimal in the sense that are very few of terms are required to capture the energy.

For flows which are, periodic in stream-wise direction x, homogenous in cross-stream direction z and inhomogeneous bounded in wall-normal direction y. Fourier transforming equations (3) in x and z directions, space correlation tensor $R_{ij}(x,x',y,y',z,z')$ becomes cross spectra tensor $S_{ij}(y,y',k_1,k_3)$ and equations (3) can be rewritten as follows:

$$\int S_{ij_{k_1k_3}}(y,y')\phi_{j_{k_1k_3}}(y')dy' = \lambda_{k_1k_3}\phi_{i_{k_1k_3}}(y) \qquad (7)$$

Solving these equations numerically, if S_{ij} is known, for each pair of wave numbers yields POD eigenfunctions $\phi_i(\cdot)$ and eigenspectra λ.

3. GOVERNING EQUATIONS

Momentum equations:

$$\frac{\partial\tilde{u}_i}{\partial t} + \tilde{u}_j\frac{\partial\tilde{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial\tilde{p}}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j^2} \qquad (8)$$

The equation for the averaged momentum:

$$\frac{\partial U_i}{\partial t} + U_j\frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu\frac{\partial^2 U_i}{\partial x_j^2} - \frac{\partial\overline{u_iu_j}}{\partial x_j} \qquad (9)$$

Fluctuation equations:

$$\frac{\partial u_i}{\partial t} + U_j\frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu\frac{\partial^2 u_i}{\partial x_j^2} - u_j\frac{\partial U_i}{\partial x_j} - u_j\frac{\partial u_i}{\partial x_j} + \frac{\partial\overline{u_iu_j}}{\partial x_j} \qquad (10)$$

Two-point Correlation Equations:

Multiplying equations (10) by u'_k and averaging yields:

$$\overline{u'_k\frac{\partial u_i}{\partial t}} + U_j\overline{u'_k\frac{\partial u_i}{\partial x_j}} = -\frac{1}{\rho}\overline{u'_k\frac{\partial p}{\partial x_i}} + \nu\overline{u'_k\frac{\partial^2 u_i}{\partial x_j^2}} - \overline{u'_ku_j}\frac{\partial U_i}{\partial x_j} - \overline{u'_ku_j\frac{\partial u_i}{\partial x_j}} \qquad (11)$$

Rewriting equations (10) for the free index (k), multiplying by u_i and averaging yields:

$$\overline{u_i\frac{\partial u'_k}{\partial t}} + U'_j\overline{u_i\frac{\partial u'_k}{\partial x'_j}} = -\frac{1}{\rho}\overline{u_i\frac{\partial p'}{\partial x'_k}} + \nu\overline{u_i\frac{\partial^2 u'_k}{\partial x'^2_j}} - \overline{u_iu'_j}\frac{\partial U'_k}{\partial x'_j} - \overline{u_iu'_j\frac{\partial u'_k}{\partial x'_j}} \qquad (12)$$

Since u_i is independent of x'_j and u'_k is independent of x_j

$$\overline{u'_k\frac{\partial u_i}{\partial t}} + U_j\frac{\partial\overline{u_iu'_k}}{\partial x_j} = -\frac{1}{\rho}\frac{\partial\overline{pu'_k}}{\partial x_i} + \nu\frac{\partial^2\overline{u_iu'_k}}{\partial x_j^2} - \overline{u'_ku_j}\frac{\partial U_i}{\partial x_j} - \frac{\partial\overline{u_iu'_ku_j}}{\partial x_j} \qquad (13)$$

$$\overline{u_i\frac{\partial u'_k}{\partial t}} + U'_j\frac{\partial\overline{u_iu'_k}}{\partial x'_j} = -\frac{1}{\rho}\frac{\partial\overline{p'u_i}}{\partial x'_k} + \nu\frac{\partial^2\overline{u_iu'_k}}{\partial x'^2_j} - \overline{u_iu'_j}\frac{\partial U'_k}{\partial x'_j} - \frac{\partial\overline{u_iu'_ku'_j}}{\partial x'_j} \qquad (14)$$

Equations (13) and (14) can be added together to yield an equation for the two-point correlation as follows:

$$\frac{\partial\overline{u_iu'_k}}{\partial t} + U_j\frac{\partial\overline{u_iu'_k}}{\partial x_j} + U'_j\frac{\partial\overline{u_iu'_k}}{\partial x'_j} = -\frac{1}{\rho}\frac{\partial\overline{pu'_k}}{\partial x_i} - \frac{1}{\rho}\frac{\partial\overline{p'u_i}}{\partial x'_k} + \nu\frac{\partial^2\overline{u_iu'_k}}{\partial x_j^2} + \nu\frac{\partial^2\overline{u_iu'_k}}{\partial x'^2_j} - $$
$$-\overline{u'_ku_j}\frac{\partial U_i}{\partial x_j} - \overline{u_iu'_j}\frac{\partial U'_k}{\partial x'_j} - \frac{\partial\overline{u_iu'_ku_j}}{\partial x_j} - \frac{\partial\overline{u_iu'_ku'_j}}{\partial x'_j} \qquad (15)$$

Replacing the velocity correlations $\overline{u_iu'_k}$ by R_{ik} and velocity-pressure correlations $\overline{pu'_k}$, $\overline{p'u_i}$ by R_{4k}, R_{i4} yields:

$$\frac{\partial R_{ik}}{\partial t}+U_j\frac{\partial R_{ik}}{\partial x_j}+U'_j\frac{\partial R_{ik}}{\partial x'_j}=-\frac{1}{\rho}\frac{\partial R_{4k}}{\partial x_i}-\frac{1}{\rho}\frac{\partial R_{i4}}{\partial x'_k}+\nu\frac{\partial^2 R_{ik}}{\partial x_j^2}+\nu\frac{\partial^2 R_{ik}}{\partial x'^2_j}-R_{jk}\frac{\partial U_i}{\partial x_j}-R_{i,j}\frac{\partial U'_k}{\partial x'_j}-\frac{\partial R_{ij,k}}{\partial x_j}-\frac{\partial R_{i,jk}}{\partial x'_j} \tag{16}$$

For a fully developed channel flow, scaling equations (16) above by u_*, ρ, and ν gives:

$$\frac{\partial R_{ik}}{\partial t}+U\frac{\partial R_{ik}}{\partial x}+U'\frac{\partial R_{ik}}{\partial x'}=-\frac{\partial R_{4k}}{\partial x_i}-\frac{\partial R_{i4}}{\partial x'_k}+\frac{1}{Re_\tau}\frac{\partial^2 R_{ik}}{\partial x_j^2}+\frac{1}{Re_\tau}\frac{\partial^2 R_{ik}}{\partial x'^2_j}-R_{2k}\frac{\partial U}{\partial y}\delta_{i1}-R_{i2}\frac{\partial U'}{\partial y'}\delta_{k1}-\frac{\partial R_{ij,k}}{\partial x_j}-\frac{\partial R_{i,jk}}{\partial x'_j} \tag{17}$$

The flow is homogenous in x and z, this means that the two-point moments can only depend on $r_1=x'-x$ and $r_3=z'-z$. Define new variables; $\xi_1=x'+x$ and $\xi_3=z'+z$, then change variables from (x',x) to (ξ_1,r_1) and from (z',z) to (ξ_3,r_3). The chain-rule implies that:

$$\frac{\partial}{\partial x'}=\frac{\partial}{\partial \xi_1}+\frac{\partial}{\partial r_1} \tag{18}$$

And

$$\frac{\partial}{\partial x}=\frac{\partial}{\partial \xi_1}-\frac{\partial}{\partial r_1} \tag{19}$$

Similarly, it is easy to show that:

$$\frac{\partial}{\partial z'}=\frac{\partial}{\partial \xi_3}+\frac{\partial}{\partial r_3} \tag{20}$$

And

$$\frac{\partial}{\partial z}=\frac{\partial}{\partial \xi_3}-\frac{\partial}{\partial r_3} \tag{21}$$

Equations (17) become:

$$\frac{\partial R_{ik}}{\partial t}+\frac{\partial}{\partial r_1}\left[U'R_{ik}-UR_{ik}\right]=\frac{\partial}{\partial r_1}\left[R_{4k}\delta_{i1}-R_{i4}\delta_{k1}\right]+\frac{\partial}{\partial r_3}\left[R_{4k}\delta_{i3}-R_{i4}\delta_{k3}\right]-\frac{\partial R_{4k}}{\partial y}\delta_{i2}-\frac{\partial R_{i4}}{\partial y'}\delta_{k2}+\frac{2}{Re_\tau}\frac{\partial^2 R_{ik}}{\partial r_1^2}+\frac{2}{Re_\tau}\frac{\partial^2 R_{ik}}{\partial r_3^2}+\frac{1}{Re_\tau}\frac{\partial^2 R_{ik}}{\partial y^2}+\frac{1}{Re_\tau}\frac{\partial^2 R_{ik}}{\partial y'^2}-R_{2k}\frac{\partial U}{\partial y}\delta_{i1}-R_{i2}\frac{\partial U'}{\partial y'}\delta_{k1}+\frac{\partial}{\partial r_1}\left[R_{i1,k}-R_{i,1k}\right]+\frac{\partial}{\partial r_3}\left[R_{i3,k}-R_{i,3k}\right]-\frac{\partial R_{i2,k}}{\partial y}-\frac{\partial R_{i,2k}}{\partial y'} \tag{22}$$

Fourier transforming in horizontal directions

$$\frac{\partial S_{ik}}{\partial t}+\imath k_1\left[U'S_{ik}-US_{ik}\right]=\imath k_1\left[S_{4k}\delta_{i1}-S_{i4}\delta_{k1}\right]+\imath k_3\left[S_{4k}\delta_{i3}-S_{i4}\delta_{k3}\right]-\frac{\partial S_{4k}}{\partial y}\delta_{i2}-\frac{\partial S_{i4}}{\partial y'}\delta_{k2}-\frac{2k_1^2}{Re_\tau}S_{ik}-\frac{2k_3^2}{Re_\tau}S_{ik}+\frac{1}{Re_\tau}\frac{\partial^2 S_{ik}}{\partial y^2}+\frac{1}{Re_\tau}\frac{\partial^2 S_{ik}}{\partial y'^2}-S_{2k}\frac{\partial U}{\partial y}\delta_{i1}-S_{i2}\frac{\partial U'}{\partial y'}\delta_{k1}+\imath k_1\left[S_{i1,k}-S_{i,1k}\right]+\imath k_3\left[S_{i3,k}-S_{i,3k}\right]-\frac{\partial S_{i2,k}}{\partial y}-\frac{\partial S_{i,2k}}{\partial y'} \tag{23}$$

Equations (23) is not valid when $y=y'$. However, one can drive a valid equation for this case starting from equations (11) and (12) above as follows;
Since u_i is independent of $x'\&z'$ and u'_k is independent of $x\&z$, equations (11) and (12) become:

$$\overline{u'_k\frac{\partial u_i}{\partial t}}+U_j\frac{\partial\overline{u_iu'_k}}{\partial x_j}(1-\delta_{j2})+V\overline{u'_k\frac{\partial u_i}{\partial y}}=-\frac{1}{\rho}\frac{\partial\overline{pu'_k}}{\partial x_i}(1-\delta_{i2})-\frac{1}{\rho}\overline{u'_k\frac{\partial p}{\partial y}}\delta_{i2}+\nu\frac{\partial^2\overline{u_iu'_k}}{\partial x_j^2}(1-\delta_{j2})+\nu\overline{u'_k\frac{\partial^2 u_i}{\partial y^2}}-\overline{u'_ku_j}\frac{\partial U_i}{\partial x_j}-\frac{\partial\overline{u_iu'_ku_j}}{\partial x_j}(1-\delta_{j2})-\overline{u'_k\frac{\partial u_iv}{\partial y}} \tag{24}$$

$$\overline{u_i\frac{\partial u'_k}{\partial t}}+U'_j\frac{\partial\overline{u_iu'_k}}{\partial x'_j}(1-\delta_{j2})+V'\overline{u_i\frac{\partial u'_k}{\partial y}}=-\frac{1}{\rho}\frac{\partial\overline{p'u_i}}{\partial x'_k}(1-\delta_{k2})-\frac{1}{\rho}\overline{u_i\frac{\partial p'}{\partial y}}\delta_{k2}+\nu\frac{\partial^2\overline{u_iu'_k}}{\partial x'^2_j}(1-\delta_{j2})+\nu\overline{u_i\frac{\partial^2 u'_k}{\partial y^2}}-\overline{u_iu'_j}\frac{\partial U'_k}{\partial x'_j}-\frac{\partial\overline{u_iu'_ku'_j}}{\partial x'_j}(1-\delta_{j2})-\overline{u_i\frac{\partial u'_kv'}{\partial y}} \tag{25}$$

Equations (24) and (25) can be added together to yield an equation for the two-point correlation as follows:

$$\frac{\partial R_{ik}}{\partial t}+U_j\frac{\partial R_{ik}}{\partial x_j}(1-\delta_{j2})+V\overline{u'_k\frac{\partial u_i}{\partial y}}+U'_j\frac{\partial R_{ik}}{\partial x'_j}(1-\delta_{j2})+V'\overline{u_i\frac{\partial u'_k}{\partial y}}==-\frac{1}{\rho}\frac{\partial R_{4k}}{\partial x_i}(1-\delta_{i2})-\frac{1}{\rho}\overline{u'_k\frac{\partial p}{\partial y}}\delta_{i2}-\frac{1}{\rho}\frac{\partial R_{i4}}{\partial x'_k}(1-\delta_{k2})-\frac{1}{\rho}\overline{u_i\frac{\partial p'}{\partial y}}\delta_{k2}+\nu\frac{\partial^2 R_{ik}}{\partial x_j^2}(1-\delta_{j2})+\nu\overline{u'_k\frac{\partial^2 u_i}{\partial y^2}}+\nu\frac{\partial^2 R_{ik}}{\partial x'^2_j}(1-\delta_{j2})+\nu\overline{u_i\frac{\partial^2 u'_k}{\partial y^2}}-R_{jk}\frac{\partial U_i}{\partial x_j}-R_{ij}\frac{\partial U'_k}{\partial x'_j}-\frac{\partial R_{ij,k}}{\partial x_j}(1-\delta_{j2})-\overline{u'_k\frac{\partial u_iv}{\partial y}}-\frac{\partial R_{i,jk}}{\partial x'_j}(1-\delta_{j2})-\overline{u_i\frac{\partial u'_kv'}{\partial y}} \tag{26}$$

For a fully developed channel flow:

$$\begin{aligned}\frac{\partial R_{ik}}{\partial t} + U\frac{\partial R_{ik}}{\partial x} + U'\frac{\partial R_{ik}}{\partial x'} &= -\frac{\partial R_{4k}}{\partial x_i}(1-\delta_{i2}) - \overline{u'_k\frac{\partial p}{\partial y}}\delta_{i2} - \\ &\quad -\frac{\partial R_{i4}}{\partial x'_k}(1-\delta_{k2}) - \overline{u_i\frac{\partial p'}{\partial y}}\delta_{k2} + \\ &+\frac{1}{Re_\tau}\frac{\partial^2 R_{ik}}{\partial x_j^2}(1-\delta_{j2}) + \frac{1}{Re_\tau}\overline{u'_k\frac{\partial^2 u_i}{\partial y^2}} + \frac{1}{Re_\tau}\frac{\partial^2 R_{ik}}{\partial x'^2_j}(1-\delta_{j2}) + \\ &+\frac{1}{Re_\tau}\overline{u_i\frac{\partial^2 u'_k}{\partial y^2}} - R_{2k}\frac{\partial U}{\partial y}\delta_{i1} - R_{i2}\frac{\partial U'}{\partial y'}\delta_{k1} - \\ &-\frac{\partial R_{ij,k}}{\partial x_j}(1-\delta_{j2}) - \overline{u'_k\frac{\partial u_i v}{\partial y}} - \frac{\partial R_{i,jk}}{\partial x'_j}(1-\delta_{j2}) - \overline{u_i\frac{\partial u'_k v'}{\partial y}}\end{aligned} \tag{27}$$

The flow is homogenous in x and z. This means that the two-point moments can only depend on $r_1 = x' - x$ and $r_3 = z' - z$.

$$\begin{aligned}\frac{\partial R_{ik}}{\partial t} + \frac{\partial}{\partial r_1}\left[U'R_{ik} - UR_{ik}\right] &= \frac{\partial}{\partial r_1}\left[R_{4k}\delta_{i1} - R_{i4}\delta_{k1}\right] + \frac{\partial}{\partial r_3}\left[R_{4k}\delta_{i3} - R_{i4}\delta_{k3}\right] - \\ &\quad -\overline{u'_k\frac{\partial p}{\partial y}}\delta_{i2} - \overline{u_i\frac{\partial p'}{\partial y}}\delta_{k2} + \\ &+\frac{2}{Re_\tau}\frac{\partial^2 R_{ik}}{\partial r_1^2} + \frac{2}{Re_\tau}\frac{\partial^2 R_{ik}}{\partial r_3^2} + \frac{1}{Re_\tau}\overline{u'_k\frac{\partial^2 u_i}{\partial y^2}} + \frac{1}{Re_\tau}\overline{u_i\frac{\partial^2 u'_k}{\partial y^2}} - \\ &\quad -R_{2k}\frac{\partial U}{\partial y}\delta_{i1} - R_{i2}\frac{\partial U'}{\partial y'}\delta_{k1} + \\ &+\frac{\partial}{\partial r_1}\left[R_{i1,k} - R_{i,1k}\right] + \frac{\partial}{\partial r_3}\left[R_{i3,k} - R_{i,3k}\right] - \overline{u'_k\frac{\partial u_i v}{\partial y}} - \overline{u_i\frac{\partial u'_k v'}{\partial y}}\end{aligned} \tag{28}$$

Rearranging the terms yields:

$$\begin{aligned}\frac{\partial R_{ik}}{\partial t} + \frac{\partial}{\partial r_1}\left[U'R_{ik} - UR_{ik}\right] &= \frac{\partial}{\partial r_1}\left[R_{4k}\delta_{i1} - R_{i4}\delta_{k1}\right] + \frac{\partial}{\partial r_3}\left[R_{4k}\delta_{i3} - R_{i4}\delta_{k3}\right] - \\ &\quad -\overline{u'_k\frac{\partial p}{\partial y}}\delta_{i2} - \overline{u_i\frac{\partial p'}{\partial y}}\delta_{k2} + \\ &+\frac{2}{Re_\tau}\frac{\partial^2 R_{ik}}{\partial r_1^2} + \frac{2}{Re_\tau}\frac{\partial^2 R_{ik}}{\partial r_3^2} + \frac{1}{Re_\tau}\frac{\partial^2 R_{ik}}{\partial y^2} - \frac{2}{Re_\tau}\frac{\partial u_i}{\partial y}\frac{\partial u'_k}{\partial y} - \\ &\quad -R_{2k}\frac{\partial U}{\partial y}\delta_{i1} - R_{i2}\frac{\partial U'}{\partial y'}\delta_{k1} + \\ &+\frac{\partial}{\partial r_1}\left[R_{i1,k} - R_{i,1k}\right] + \frac{\partial}{\partial r_3}\left[R_{i3,k} - R_{i,3k}\right] - \overline{u'_k\frac{\partial u_i v}{\partial y}} - \overline{u_i\frac{\partial u'_k v'}{\partial y}}\end{aligned} \tag{29}$$

Fourier transforming in horizontal directions:

$$\begin{aligned}\frac{\partial S_{ik}}{\partial t} + \imath k_1\left[U'S_{ik} - US_{ik}\right] &= \imath k_1\left[S_{4k}\delta_{i1} - S_{i4}\delta_{k1}\right] + \imath k_3\left[S_{4k}\delta_{i3} - S_{i4}\delta_{k3}\right] - \\ &\quad -\overline{\hat{\hat{u}}'_k\frac{\partial\hat{\hat{p}}}{\partial y}}\delta_{i2} - \overline{\hat{\hat{u}}_i\frac{\partial\hat{\hat{p}}'}{\partial y}}\delta_{k2} - \\ &\quad -\frac{2k_1^2}{Re_\tau}S_{ik} - \frac{2k_3^2}{Re_\tau}S_{ik} + \frac{1}{Re_\tau}\frac{\partial^2 S_{ik}}{\partial y^2} - \frac{2}{Re_\tau}\overline{\frac{\partial\hat{\hat{u}}_i}{\partial y}\frac{\partial\hat{\hat{u}}'_k}{\partial y}} - \\ &\quad -S_{2k}\frac{\partial U}{\partial y}\delta_{i1} - S_{i2}\frac{\partial U'}{\partial y'}\delta_{k1} + \\ &+\imath k_1\left[S_{i1,k} - S_{i,1k}\right] + \imath k_3\left[S_{i3,k} - S_{i,3k}\right] - \overline{\hat{\hat{u}}'_k\frac{\partial\hat{\hat{u}}_i\hat{\hat{v}}}{\partial y}} - \overline{\hat{\hat{u}}_i\frac{\partial\hat{\hat{u}}'_k\hat{\hat{v}}'}{\partial y}}\end{aligned} \tag{30}$$

Continuity equation implies that:

$$\frac{\partial S_{2k}}{\partial y} + \frac{\partial S_{i2}}{\partial y'} = \imath k_1[S_{1k} - S_{i1}] + \imath k_3[S_{3k} - S_{i3}] \tag{31}$$

The doubly Fourier transformed random velocity component, $\hat{\hat{u}}_i(y,t)$, can be reconstructed from the POD eigenfunctions as follows:

$$\hat{\hat{u}}_i(y,t) = \sum_{n=1}^{\infty} a^{(n)}(t)\phi_i^{(n)}(y) \tag{32}$$

The random coefficients $a^{(n)}$ are uncorrelated and POD eigenfunctions $\phi_i^{(n)}$ are orthogonal.

$$S_{ik}(y,y') = \overline{\hat{\hat{u}}_i(y)\hat{\hat{u}}_k^*(y')} \tag{33}$$

$$S_{ik}(y,y') = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\overline{a^{(n)}a^{(m)*}}\phi_i^{(n)}(y)\phi_k^{(m)*}(y') \tag{34}$$

$$S_{ik}(y,y') = \sum_{n=1}^{\infty}\lambda^{(n)}\phi_i^{(n)}(y)\phi_k^{(n)*}(y') \tag{35}$$

$$\lambda^{(n)} = \overline{a^{(n)}a^{(m)*}}\delta_{nm} \tag{36}$$

4. CLOSURE MODELS

Equations (23), (30) and (31) form a set of equations that can be solved to obtain the two-point cross-spectra terms, S_{ik}, for each pair of wave-numbers (k_1, k_3). However, these equations are not closed, i.e., the unknowns are more than the equations we have. The set of equations can be closed using the following closure solutions:

$$S_{ij,k}(y,y') = \overline{\hat{\hat{u}}_i(y)\hat{\hat{u}}_j(y)\hat{\hat{u}}_k^*(y')} \tag{37}$$

$$S_{ij,k}(y,y') = \sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\overline{a^{(n)}a^{(l)}a^{(m)*}}\phi_i^{(n)}(y)\phi_j^{(l)}(y)\phi_k^{(m)*}(y') \tag{38}$$

$$S_{ij,k}(y,y') \propto \sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sqrt{\lambda^{(n)}\lambda^{(l)}\lambda^{(m)}}\phi_i^{(n)}(y)\phi_j^{(l)}(y)\phi_k^{(m)*}(y') \tag{39}$$

$$S_{ij,k}(y,y') = \sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\Lambda^{(n,l,m)}\phi_i^{(n)}(y)\phi_j^{(l)}(y)\phi_k^{(m)*}(y') \tag{40}$$

$$S_{i,jk}(y,y') = \overline{\hat{\hat{u}}_i(y)\hat{\hat{u}}_j^*(y')\hat{\hat{u}}_k^*(y')} \tag{41}$$

$$S_{i,jk}(y,y') = \sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\overline{a^{(n)}a^{(l)*}a^{(m)*}}\phi_i^{(n)}(y)\phi_j^{(l)*}(y')\phi_k^{(m)*}(y') \tag{42}$$

$$S_{i,jk}(y,y') \propto \sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sqrt{\lambda^{(n)}\lambda^{(l)}\lambda^{(m)}}\phi_i^{(n)}(y)\phi_j^{(l)*}(y')\phi_k^{(m)*}(y') \tag{43}$$

$$S_{i,jk}(y,y') = \sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \Lambda^{(n,l,m)*} \phi_i^{(n)}(y) \phi_j^{(l)*}(y') \phi_k^{(m)*}(y') \quad (44)$$

$$|\overline{a^{(n)}a^{(l)}a^{(m)*}}| = |\overline{a^{(n)}a^{(l)*}a^{(m)*}}| = \sqrt{\lambda^{(n)}\lambda^{(l)}\lambda^{(m)}} \quad (45)$$

$$\Lambda^{(n,l,m)} = \sqrt{\lambda^{(n)}\lambda^{(l)}\lambda^{(m)}} e^{\imath\Theta^{(n,l,m)}} \quad (46)$$

$$\overline{\frac{\partial \hat{u}_i}{\partial y}\frac{\partial \hat{u}'_k}{\partial y}} = \sum_{n=1}^{\infty} \lambda^{(n)} \frac{\partial \phi_i^{(n)}}{\partial y} \frac{\partial \phi_k^{(n)*}}{\partial y} \quad (47)$$

$$\overline{\hat{u}'_k \frac{\partial \hat{\hat{u}}_i \hat{\hat{v}}}{\partial y}} \simeq \sum_{n=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \Lambda^{(n,l,m)} \frac{\partial \phi_i^{(n)} \phi_2^{(l)}}{\partial y} \phi_k^{(m)*} \quad (48)$$

$$\overline{\hat{u}'_k \frac{\partial \hat{\hat{p}}}{\partial y}} \simeq \sum_{n=1}^{\infty} \lambda^{(n)} \frac{\partial \phi_4^{(n)}}{\partial y} \phi_k^{(n)*} \quad (49)$$

5. NUMERICAL SOLUTION

Fig. 1 shows a sketch of the flow geometry. The flow is driven by constant pressure gradient in the stream-wise direction which is balanced by wall friction stress on both walls. Therefore, the flow is homogeneous in the stream-wise and cross-stream directions. In additions, the statistics are dependent only on the distance from the wall. Since the flow is stationary in time, time derivatives are ignored. The computational domain was chosen to be $15.3h \times 2h \times 7.68h$. Where is similar to that of DNS, and divided into $256 \times 129 \times 256$ grid points, in the stream-wise, wall-normal, and cross-stream directions, respectively. The size of the domain and the grid distribution are used to determine the wave-numbers (k_1, k_3) in the stream-wise and cross-stream directions.

Since the flow is symmetric in the wall-normal direction, the computation was carried out for half of the channel, from one wall to the channel center line.

The following boundary conditions were used at the channel center line:

$$\frac{\partial S_{11}}{\partial y} = 0, \quad S_{12} = 0, \quad \frac{\partial S_{13}}{\partial y} = 0, \quad S_{21} = 0$$

$$\frac{\partial S_{22}}{\partial y} = 0, \quad S_{23} = 0, \quad \frac{\partial S_{31}}{\partial y} = 0, \quad S_{32} = 0$$

$$\frac{\partial S_{33}}{\partial y} = 0$$

The following symmetry conditions were used:

$$S_{21} = {S_{12}}^T, \quad S_{31} = {S_{13}}^T, \quad S_{32} = {S_{23}}^T$$

$$S_{41} = {S_{14}}^T, \quad S_{42} = {S_{24}}^T, \quad S_{43} = {S_{34}}^T$$

Where, the superscript T denotes transpose.

Equations (23), (30) and (31) were discretized and solved for each pair of wave-numbers, assuming that the mean velocity profile is known, using the following algorithm:

1. Initial guess of eigenfunctions (ϕ_1, ϕ_2, ϕ_3) and eigenvalue (λ).
2. $S_{ik}, \Lambda, S_{i,jk}, S_{ij,k}$ and other closure terms are calculated utilizing the eigenfunctions (ϕ_1, ϕ_2, ϕ_3), eigenvalue (λ) and symmetry conditions.
3. Equations (23), (30) and (31) are solved for S_{ik} and S_{i4}.
4. The obtained S_{ik} from step 3 is used to solve POD eigenvalue problem, equation (7), and to get a new set of eigenfunctions (ϕ_1, ϕ_2, ϕ_3) and eigenvalue (λ).
5. Steps 2, 3 and 4 are repeated till convergence.

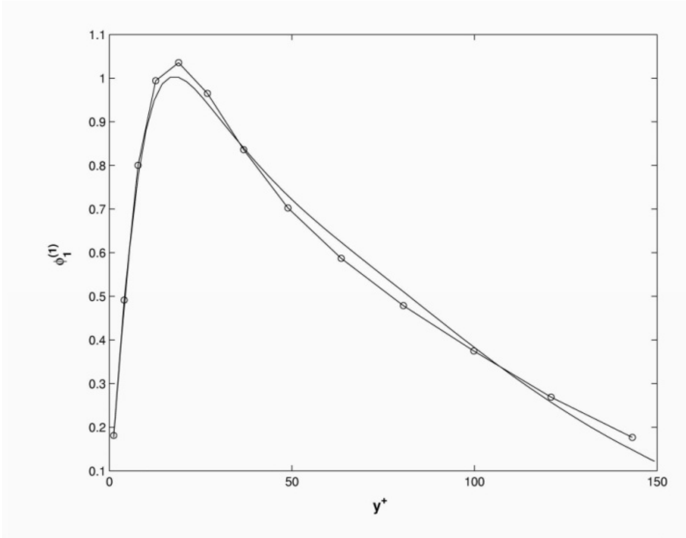

Fig. 2. Stream-wise POD eigenfunction component, $-o-$ extracted from DNS data,— calculated using current method.

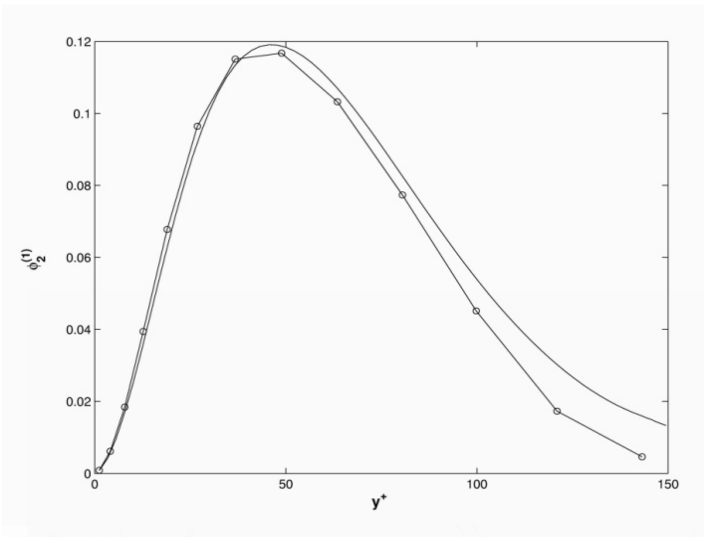

Fig. 3. Wall-normal POD eigenfunction component, $-o-$ extracted from DNS data,— calculated using current method.

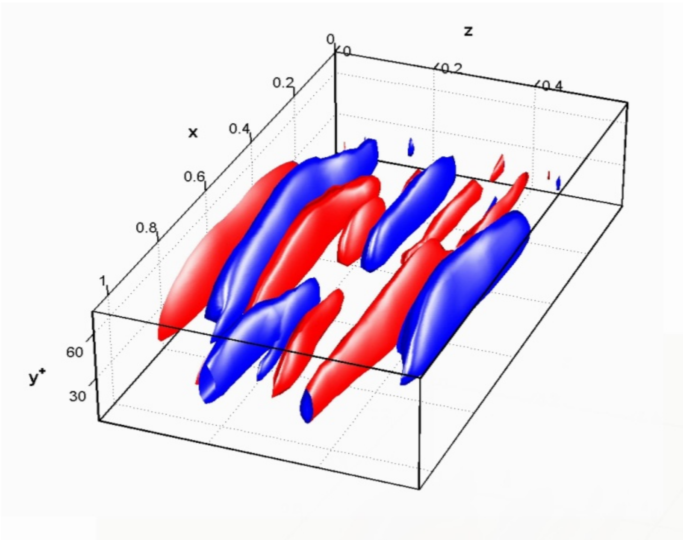

Fig. 4. Isosurface of stream-wise vorticity reconstructed using POD eigenfunctions that was extracted from DNS data, red $+ve$ and blue $-ve$.

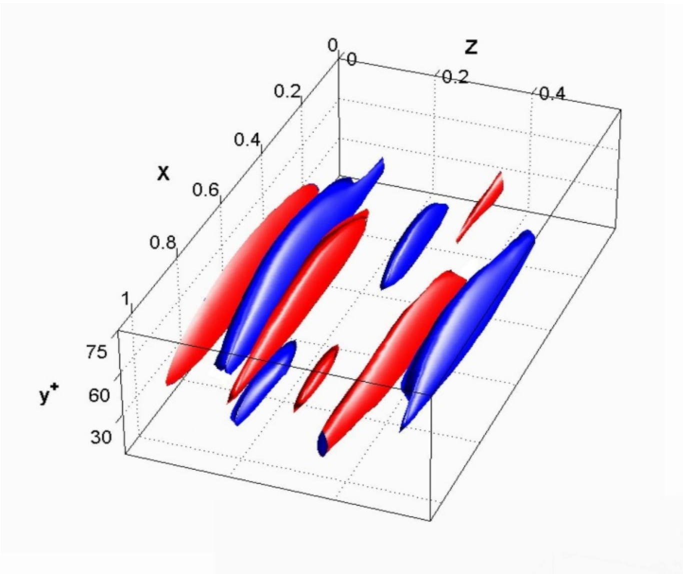

Fig. 5. Isosurface of stream-wise vorticity reconstructed using POD eigenfunctions that was calculated using current method, red $+ve$ and blue $-ve$.

6. CONCLUSIONS

In this paper we have developed an analytical procedure for extracting basis functions which approximate those given by POD method (POD eigenfunctions). The POD method requires cross-spectral tensor (S_{ij}) as the kernel of its eigenvalue problem. Thus necessitating complete information of the flow before the analysis can proceed. For flows with very high Reynolds number, it can be expensive if not impossible with the current computational and experimental capabilities. However, in this method the cross-spectral tensor (S_{ij}) is determined from the solution of Fourier transformed two-point correlation equations. POD eigenfuctions that was extracted from DNS data and the basis functions that was calculated using the current method for stream-wise (ϕ_1) and wall-normal (ϕ_2) are similar to each other as shown in Figs 2 and 3 above. However, there are some discrepancies away from the wall. These discrepancies are due to the energy transfer model, $S_{i,jk}$, especially the cross-stream component. Reconstruction of near wall coherent structures using calculated basis functions have given smoother structures compared to those reconstructed using POD eigenfunctions as shown in Figs 4 and 5. Again, this is due to the underestimation of the cross-stream energy transfer model. This method can be further developed to be a model for turbulence by solving the mean momentum equation iteratively for the model input (mean velocity).

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