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On The Integration and Evaluation of Vertical Control Information and Uncertainties in Leveling Networks Using Least Squares Modeling

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Abstract: Proper integration and evaluation of an existing vertical control information with the adjustment of a new leveling network require a stepwise approach that could reveal the hidden aspects of their uncertainties or stochastic properties. The general use of the control information as fixed quantities in the adjustment of the leveling networks is a major flaw. To this end, the fundamental concepts of least squares solutions offer a flexible and a rich framework for proper integration and modeling of control information and their uncertainty for new leveling networks. This paper provides a comprehensive review and analysis of a workflow that can be used to integrate and evaluate the existing control information or benchmarks to a new leveling network. In particular, this paper exploits three different approaches of least squares solutions to integrate and evaluate the stochastic properties of the existing control information and observations that belong to a new network. First, ordinary least squares solution, which constrained by Gauss-Markov model, was exploited to depict the normal practice of leveling networks adjustment in which the control information will be introduced as constant or fixed values. Second, least squares solution with pseudo observations was exploited for proper integration of control information and their stochastic properties. Third, free-network least squares solution was exploited as a mechanism to separate and quantify the stochastic properties of the observations from the ones that will be associated with the control information. Through the use of a numerical example, this paper offers some new perspectives and a detailed analysis that explains the interplay between the different aspects of least squares solutions for the integration and evaluation of vertical control information and their uncertainties with new leveling networks.

Keywords: *Leveling Networks; Control Information; Uncertainty; Least Squares Modeling*

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1. INTRODUCTION

Although the adjustment of leveling networks is a classical topic and practice in the surveying works, proper integration and evaluation of an existing vertical control information or the values of benchmarks to a new network is not a trivial task and is not deeply understood. Surprisingly enough, this lack of understanding exists at the conceptual and the practical levels. In one hand, this lack of understanding may be explained by the way in which the subject of adjustment computations was taught. On the other hand, this lack of understanding may be explained by the weak relationship between the professional practice and the advanced concepts of modeling uncertainties in the general framework of least squares solutions. For example, handling or integrating the control information as fixed values to a new leveling network will ignore their uncertainties or stochastic properties. Specifically, it will ignore the stochastic interplay between the new observations and the existing control information in terms of exchanging the benefit of accuracy improvement among each other. As such, the new leveling network could be viewed as a local or isolated network since it does not acquire the stochastic properties of the existing benchmark/s. In other words, the new network will not be correctly tied or unified with the existing control information or network and it will miss the opportunity of proper information update and integration. This is equally true for adjustment by observation and condition equations. In fact, this is more true and obvious for the condition equations since their formulation are completely dependent on local constraints between the observations that do not include any knowledge about the existing control information. This is not the first paper to address this issue. For example, more than two decades ago Schwarz [1] addressed the same issue in the context of GPS network adjustment and update, which is very similar to the issue of the leveling network, which will be addressed in this paper.

As is well known and from theoretical proofs, proper weighting of the least squares solution of a leveling network is inversely proportional to the distances between the leveling points or the height difference between pairs of points [2]. In other words, the uncertainties or the relative contribution of different observations in the least squares solution of the leveling networks are expressed in terms of distances. Sometimes the uncertainties of the existing vertical control information are expressed in terms of their standard deviations or variances, which were estimated from their previous dispersion or variance-covariance matrices. Although this type of expression or representation of uncertainties is very common, it makes the integration of the uncertainties of the existing vertical control information to a new leveling network a non-trivial task and creates a numerical imbalance among the elements of the weight matrix that will be used to solve for the parameters of the new leveling network. In particular, this representation may generate a very large reference variance or variance component, which may give a wrong message or indication about the quality of the obtained solution from the least squares in terms of its global fit. Therefore, the standard deviations of the control information, which were obtained from a previous adjustment, should be transformed to their equivalent distances for proper inclusion or integration to the least squares solution of new leveling networks. In addition and as a recommendation, the representation of the uncertainties of the benchmark values should be extended to include the standard deviations as well as the distances that were used to build their previous weighting matrices.

This paper argues that proper integration and evaluation of an existing vertical control information with the adjustment of a new leveling network require a stepwise approach that could reveal the hidden aspects of their uncertainties or stochastic properties. To this end, the fundamental concepts of least squares solutions offer a flexible and a rich framework for proper integration and modeling of control information and their uncertainty for new leveling networks. This paper provides a comprehensive review and analysis of a workflow that can be used to integrate and evaluate the control information or benchmarks values to new leveling networks. In particular, this paper exploits three different approaches of least squares solutions to integrate and evaluate the stochastic properties of the control information and observations of a new leveling network. First, ordinary least squares solution, which will be constrained by linear Gauss-Markov model, was exploited to depict the normal practice of leveling networks adjustment in which the control information will be introduced as constants or fixed values. In other words, the first approach ignores the stochastic properties of the control information and leaves the new leveling network defined in a local vertical datum or isolated from the existing network. More importantly, the first approach will serve as a baseline for comparison with the other two approaches. Second, least squares solution with pseudo observations was exploited for proper integration of control information and their stochastic properties to a new leveling network. The second approach offers a very elegant framework for the inclusion of the stochastic properties of the control information as well as the ones of new observations in one unified framework that mimic the original Gauss-Markov model. Moreover, it avoids the special handling of the stochastic properties of the control information (Benchmarks) if their uncertainties were modeled within the normal representation of ordinary least squares solution or a modified Gauss-Markov Model with error propagation in which the uncertainties of the control information will be part of a restricted weight matrix. This weight matrix will have the same size of the given weight matrix of the observations of the new leveling network. Although there is some reservation against the least squares with pseudo observations for large data sets [3], this reservation can be handled, for example, by sequential adjustment. Third, a free-network least squares solution was exploited as a mechanism to separate and quantify the stochastic properties of the observations from the ones that will be associated with the control information. In particular, the uniqueness of the reference variance or the variance component will be used as a measure for the global fit and quality control of the stochastic properties of the observations in a new leveling network.

This paper is organized as follows. Section two provides a detailed review for the three approaches of least squares solution that will be used in the test example. Section three explains the research methodology of this work. Section four shows the test example and its relevant data that will be used to demonstrate the argument of this paper. Section five presents the results and analysis of the test example. Section six concludes the paper with some recommendations.

2. LEAST SQUARES MODELING

As stated, this paper exploits three different approaches of least squares solutions to integrate and evaluate the stochastic properties of the existing control information and new observations. Namely, it exploits ordinary least squares solution or linear Gauss-Markov model, least squares solution with pseudo observations, and Free-network least squares solution.

During the discussion in this section, the simulated leveling network shown in **Fig. 1** will be used to explain the different aspects of the three approaches of the least squares solutions. **Table 1** shows the relevant data of the simulated leveling network in terms of the observed difference in heights (y_1, ..., y_2) between points (H1, H2, H3, H4), the distances (L_1, ..., L_5) between points, and the standard deviations of each observation (σ_{y_1}, ..., σ_{y_5}).

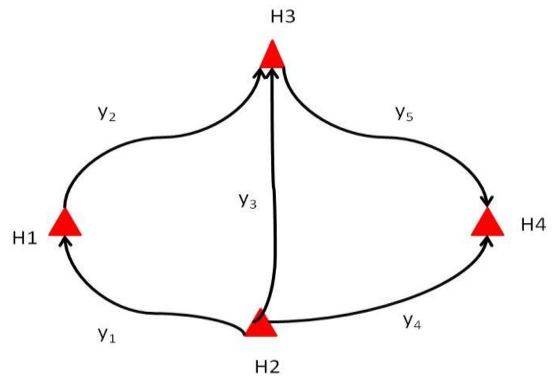

Fig. 1. A simulated leveling network

Table 1. Relevant data for the simulated example

Observation	Distance Between Point (km)	Standard Deviation
y_1	L_1	$\pm\sigma_{y_1}$
y_2	L_2	$\pm\sigma_{y_2}$
y_3	L_3	$\pm\sigma_{y_3}$
y_4	L_4	$\pm\sigma_{y_4}$
y_5	L_5	$\pm\sigma_{y_5}$

The general formulation of Gauss-Markov model that constrains the ordinary least squares solution is typically depicted by the following equation:

$$Y_{nx1} = A_{nxm}\xi_{mx1} + e_{nx1}, \quad e \sim (0, \sigma_0^2 P^{-1}) \tag{1}$$

where: Y: Observations vector (here: differences in elevations between points pairs).
n : Number of observations.
m : Number of unknowns or parameters (here: heights of points).
A : Design matrix.
ξ : Unknown parameters.
e : True error vector.
σ_0^2 : Unit reference variance.
$P_{n\,x\,n}$: A diagonal weight matrix for the observations vector (Y).

The target function for the least squares solution, which is constrained by Gauss-Markov model shown in Equation 1 is:

$$\Phi(e,\lambda,\xi) = e^T Pe + 2\lambda^T (Y - A\xi - e) \tag{2.a}$$

It should be noted that the target function in equation (2.a) is expressed in terms of Lagrange multiplier vector (λ). Practically, Lagrange multipliers offer an automatic mechanism for direct accommodation or inclusion of the constraint/s with a given function [4]. Equation (2.b) offers another representation for the target function that does not include the Lagrange multiplier vector and the error vector (*e*). In other words, it gives a direct minimization of the squared weighted error in terms of the observation vector (Y), the design matrix (A), the weight matrix (*P*) and the unknown parameters vector (ξ) and it restricts the minimization process to the parameters vector.

$$\Phi(\xi) = (Y - A\xi)^T P(Y - A\xi) \tag{2.b}$$

where: Φ : Target function to be minimized with respect to (e,λ,ξ) in 2.a or (ξ) in 2.b.
λ : Lagrange multiplier.

The manipulation of the target function shown in (2.a or 2.b) by the minimization process will lead to the following set of equations:

$$\hat{\xi} = \underbrace{(A^T PA)^{-1}}_{N^{-1}}\underbrace{A^T PY}_{C} \tag{3}$$

$$\tilde{e} = Y - A\hat{\xi} \tag{4}$$

$$\hat{\sigma}_0^2 = \frac{\tilde{e}^T P\tilde{e}}{\underbrace{n-m}_{r}} \tag{5}$$

$$D\{\hat{\xi}\} = \sigma_0^2 N^{-1} \tag{6.a}$$

$$\hat{D}\{\hat{\xi}\} = \hat{\sigma}_0^2 N^{-1} \tag{6.b}$$

where: $\hat{\xi}$: Estimated parameters vector.
$D\{\hat{\xi}\}$: Dispersion or variance co-variance matrix of the unknown parameters.
$\tilde{e}$: Predicted residuals vector.
$\hat{\sigma}_0^2$: Estimated reference variance or variance component.
n : Number of observations or equations.
m : Number of unknown parameters or observation's equations.
r : redundancy number.
N : Normal matrix.

Equation 6.a captures the geometry in terms of the connectivity and directions of the nodes in the leveling network before applying the scaling by the estimated variance component shown in Equation 5. Therefore, it is called the a priori dispersion or variance-covariance matrix. This matrix can be computed before the commencement of the field work since its elements consists of constant values (-1, 0, 1) and an assumed variance of the observations, which can be specified with high confidence in advance. Equation (6.b) captures the geometry of the leveling network after the field work since its definition depends on the estimated reference variance or variance component from the adjustment process.

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{bmatrix}}_{\xi} + e_{5x1} \tag{7}$$

It is very simple to observe that the design matrix ***A*** has a rank deficiency or dependency of one, which can be checked by adding the third and the fifth row to generate the fourth row. Let us fix the height of the fourth point (H_4) to remove the rank deficiency in the design matrix ***A***. By doing this, the fourth column in the design matrix will be removed and added to the vector of observations ***Y***. Therefore, Equation 7 can be rewritten as follows:

$$\underbrace{\begin{bmatrix} y_1 - 0 \\ y_2 - 0 \\ y_3 - 0 \\ y_4 - H_4 \\ y_5 - H_4 \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} H_1 \\ H_2 \\ H_3 \end{bmatrix}}_{\xi} + e_{5x1} \qquad (8)$$

By moving H_4 to the left hand side of Equation 8, a big assumption was made. H_4 is treated as a fixed quantity or constant value, which is a safe assumption for a local leveling network. Local networks are used for measurement in confined engineering projects such as stakeout of buildings or earthwork computations. On the other hand, this is not a correct assumption when the new network is to be tied to an existing or national leveling network. In fact, even in engineering projects that require absolute measurement for deformation, for example, in dams and bridges, the concept of local network is not an adequate one since it will not account for the stochastic properties of the previous measurements. In general, H_4 or any other point in the network could belong to an existing leveling network and their values will be associated with some figures of accuracy or standard deviations. Therefore, ignoring the stochastic properties of H_4 or any other point will render the network work shown in Figure 1 as a local network and will it not be correctly tied to the existing leveling network.

Gauss-Markov model with pseudo-observations can be used for proper integration of the stochastic properties of the existing control information. The model shown in (7) will be modified and extended to realize the given control information as a pseudo-observation in which H_4 will be treated as an unknown parameter as well as an observation. As shown in equation (9), the given value of H_4 will be an element of the observation vector ***Y***. The rank deficiency in design matrix (A) will be removed by adding a pseudo observation for H_4 as a new row in the design matrix (***A***) as shown in equation (9) and H_4 appears as an element in the unknown parameters vector (ξ). In general, the least squares solution with pseudo observations transforms the control information into unknown parameter/s and uses their prior knowledge or values to construct a stochastic constraint to overcome the rank deficiency of the design matrix (A). As such, the least squares solution with pseudo observations can be seen as a special case of the generalized inverse solution.

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ H_4^0 \end{bmatrix}}_{Y} = \underbrace{\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ H_4 \end{bmatrix}}_{\xi} + \begin{bmatrix} e_{5x1} \\ e_{H_4^0} \end{bmatrix} \qquad (9)$$

The free network solution of the least squares deals with the problem of the generalized inverse of a rank deficient matrix (***A***) [5]. Although the generalized inverse does not give a unique solution for the unknown parameters [6], the reference variance or the variance component shown in equation (6) is unique. In other words, the estimated value of the variance component does not depend on which generalized inverse that was used to estimate the unknown parameters. The uniqueness of the variance component will be exploited in this paper to separate the stochastic properties of the new observations from the ones that will be associated with the existing control information. Although the same process of errors separation can be obtained by the use of condition equations, it is beyond the scope of this paper.

The unique minimum-norm condition or inner constraints [7] can be used to get an optimal or unique least square solution. In other words, the minimum-norm solution will give the smallest standard deviations for the unknown parameters of the leveling network. The estimated parameters in the free network solution can be obtained by:

$$\hat{\xi} = (A^T PA + D^T D)^{-1} A^T PY \qquad (10)$$

The matrix D should satisfy:

$$D\hat{\xi} = 0 \qquad (11)$$

The structure of the matrix D, which contains the elements of the inner constraints, depends on the structure of the design matrix A or the geometric configuration of the leveling network. For a leveling network and as shown in Equation 11, an inner constraint forces the average heights of all unknown parameters in the network to remain unchanged or their sum is equal to zero. Mathematically, the inner constraint/s in the form of the D matrix can be derived by the use of the similarity transformation or S-transformation for short [7]. As such, it should be noted that the inner constraint/s will derived by a functional relationship that is not part of the target function of the least squares minimization process. The least squares solution only enforces the constraints shown in Equations 1 and 11. Therefore, the free network solution in the context of the minimum-norm is a two steps process. First, is to identify the structure of the minimum constraint/s. Second, is to apply the least squares minimization process. This type of clarification is very critical for proper understanding and implementation of the least squares solutions. In fact, this clarification will lead us to the differentiation between minimum constraint/s and inner constraint/s. Inner constraint/s have to do with the minimum-norm solution and this is not the case for general definition of the minimum constraint/s.

Now we turn our attention on how to convert the standard deviations of existing control points into distances. In general, the uncertainties or the relative contribution of different observations in the least squares solution of the leveling networks are expressed in terms of distances [2]. Sometimes the uncertainties of the existing control information are expressed in terms of their standard deviations, which were estimated from their previous dispersion or variance-covariance matrices. Although this type of expression or representation of uncertainties is very common, it makes the integration of the uncertainties of the control information to a new network a non-trivial task. To this end, the standard deviation of each observation in a new leveling network can be converted to its equivalent distance by the following relationship:

$$K = \left[\frac{\sigma}{m}\right]^2 \qquad (12)$$

where: K : Distance in km.
m : A constant that is directly related to the order of the survey.
σ : The given standard deviation of an existing control point in *mm*.

Table 2 shows the US Federal Geodetic Control Subcommittee (FGCS) vertical control survey accuracy standards by which we can convert the standard deviation of existing control points into distances and introduce them in the weight matrix of the least squares solution [8]. Yes indeed, this conversion will come at the cost of knowing the order and the class of the survey.

Equation 12 should be used with a very special care since its conception is based on empirical findings and it is not directly connected to the variance-covariance matrices shown in (6.a or 6.b). In other words, Equation 12 expresses the misclosure between two leveling points as a function of a distance and some constant factor (m) that encapsulates the overall ecosystem of the measurement process. In particular, Equation 12 does not capture the geometry of the leveling network, which will be expressed by the normal matrix (***N***) that was shown in Equations 6.a and 6.b. In bold terms, it has nothing to do with the geometry of the leveling network. In fact, it can be said that there is a knowledge gap that could explain the connection or the link between the empirical formula shown in Equation 12 and the theoretical accuracy shown in Equations 6.a and 6.b. This gap warrants a future investigation to generalize and validate the equivalency between the accuracy figures of points pairs that can be obtained by Equation 12 and the ones that will be encountered in the leveling network as a function of the variance-covariance matrix.

Table 2. 1984 FGCS Vertical Control Survey Accuracy Standards

Order and Class	$\pm\sigma$ (mm)
First Order: Precise Leveling	
Class I	$0.5\times\sqrt{K}$
Class II	$0.7\times\sqrt{K}$
Second Order: For Engineering Works	
Class I	$1.0\times\sqrt{K}$
Class II	$1.3\times\sqrt{K}$
Third Order	$2.0\times\sqrt{K}$

3. MATERIALS AND METHODS

The research methodology is based on a gradual testing of different scenarios on a simple example to clarify the conceptual and the practical aspects of the adjustment of leveling networks. These scenarios will be given in section five. The overall objective of these scenarios is to demonstrate the main argument of this paper, which states that:

"*Proper integration and evaluation of vertical control information with the adjustment of a new leveling network require a stepwise approach that could reveal the hidden aspects of their uncertainties or stochastic properties*"

The principles of least squares solutions that were shown in the previous section will be used in this testing. In particular, the research methodology will investigate the following issues:

- Different weighting schemes. Four different types of weighting will be tested.
- Different scenarios for the use of the control information.
- The three approaches of least squares solutions.

3.1 Test Example

Fig. 2 shows the layout of the topology of the test example for a leveling network, which consists of two known control points (BM X=100.00 m and BM Y=107.50 m), three unknown points (A, B, and C) and seven observations. This example was presented in [2] and it was chosen in this research to share a common test bed for the ideas presented in this paper. All units of this example were set on the metric system since the overall comparisons in the carried tests will be based on their relative values. Table 3 shows the relevant data for the example shown in Figure 2. It is important to note that this is an over constrained network since it uses two control points (BM X and BM Y). BM X and BM Y were assigned an identical standard deviation ($\pm$ 0.004 m) and distance (8.5 km) for the use in the weight matrix during the parameters estimation process by least squares.

In general, only one control point is needed to constrain or to remove the rank deficiency or to fix the datum problem of leveling network. The provision of one control point can be seen as a minimum external constraint in light of the free network solution. The availability of two control points will give us the opportunity to use each one of them can serve the dual role of a control point as well as a check point. This type of use will be utilized to compare internal accuracy of the least squares solution, which will be computed from variance-covariance matrix shown in Equation 6.b, with the external accuracy in terms of root-mean-square-error (*RMSE*). The *RMSE* will be obtained from the comparison between the given value of the control information (here: BM X and BM Y) and their estimated values from the least squares solution.

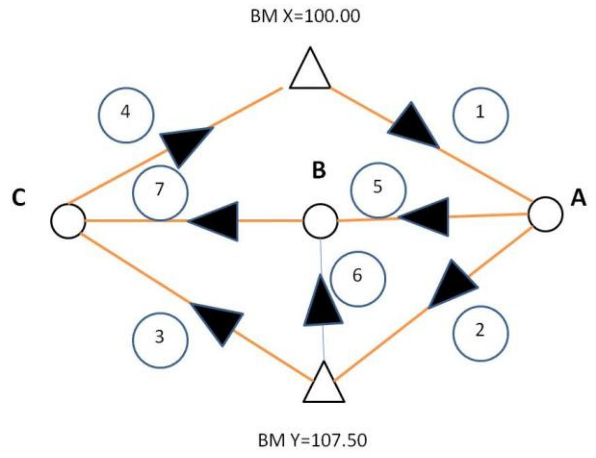

Fig. 2. Layout for the test example

Table 3. Data for the test example

Line	Observed Elevation Differences	Length (km)	Standard Deviation (m)
1	5.1000	4	± 0.006
2	2.3400	3	± 0.005
3	-1.2500	2	± 0.003
4	-6.1300	3	± 0.005
5	-0.6800	2	± 0.003
6	-3.0000	2	± 0.003
7	1.7000	2	± 0.003

4. RESULTS AND DISSCUSSION

In this work, thirteen cases were tested and analyzed. These cases were designed to cover the three approaches of the least squares solutions that were proposed in this paper. **Table 4** shows the explanation of each case. **Table 5** lists the numerical results for each case shown in **Table 4**. A MATLAB prototype software was developed to carry out the proposed tests that were outlined in **Table 4**.

Fig. 3 shows a plot of the variance components for the thirteen cases in which there are two anomalies (cases no. 3 and 10). The highest values were obtained by the use of the standard deviations in the weight matrix for cases no. 3 and 10. For more information please check **Tables 4** and **5**. Clearly, these high values suggest that the use of the standard deviations in the weight matrix is not an optimal strategy to reflect the contributions of different observations in a global model fitting in terms of the variance component.

In the first case, an un-weighted or equal weighted linear Gauss-Markov Model by an identity weight matrix was used. Except for cases no. 3 and 10, case no. 1 has the highest variance component value (0.0025), which suggests that an un-weighted or weighting by an identity matrix is a sub-optimal strategy for weighting but is not a bad choice and this is even with the availability of the uncertainties in terms of standard deviation as suggested by the discussion in cases no. 3 and 10. In case no. 2, the variance component was improved by a factor of 2.5 (0.001) since the given distances shown in Table 3 were used to construct the weight matrix. This finding complies with theoretical proofs of weighting by distances [2] between leveling points. In addition, there are no major differences between the accuracy of the estimated parameters (A, B, C) in cases no. 1 and 2, which suggests that the variance component has a very minor influence on the dispersion or the variance-covariance matrix. In fact this issue requires a deeper investigation in a separate paper to understand and analyze the impact of the weight matrix on the derived parameters and information from least squares solutions.

Although case no. 3 has the highest value of the variance component, surprisingly enough it induced an improvement in the estimated accuracy of B and C and this is in comparison with the ones that were obtained from case no. 2, which uses the optimal weighting by distances. Therefore, the notion of optimal weighting by distances is not a global one. Yes, the values of the estimated parameters were changed between cases no. 2 and 3. Ones again, this finding supports the requirement for a further investigation to understand the impact of different weighting schemes on the derived information and parameters from least squares solutions.

In case no. 4, normalized distances were used to construct the weight matrix. In particular, all distances shown in Table 3 were divided by their minimum value (2). Division by the minimum distance was used to comply with the notion of giving more weight to the smallest distance since it should contribute by a higher value in the weight matrix. The only difference between cases no. 2 and 4 is the doubling of the variance component since it went from 0.001 in case no. 2 to 0.0019 in case no. 4. On the other hand, the values of the

Table 4. Test cases

Case No.	Explanation
1	Un-weighted: linear Gauss-Markov Model
2	Weighted by distances: linear Gauss-Markov Model
3	Weighted by variance: linear Gauss-Markov Model
4	Weighted by normalized distance: linear Gauss-Markov Model
5	Un-weighted: linear Gauss-Markov Model in which BM Y is unknown
6	Un-weighted linear Gauss-Markov Model in which BM X is unknown
7	Weighted by distance: linear Gauss-Markov Model in which BM X is unknown
8	Un-weighted: linear Gauss-Markov Model with Pseudo Observations
9	Weighted by distance: linear Gauss-Markov Model with Pseudo Observations
10	Weighted by variance: linear Gauss-Markov Model with Pseudo Observations
11	Un-weighted: Free-Network Solution Using D_1 matrix
12	Un-weighted: Free-Network Solution Using D_2 matrix
13	Weighted by distances: Free-Network Solution Using D_1 matrix

estimated parameters (A, B, C) and their accuracies remain the same. This suggests that the normalization does not impact the estimated parameters and its impact was cancelled from the variance-covariance matrix. Therefore, normalization as shown in this case does not provide an advantage over the classical or the regular weighting by the pure inverse of the distances. On the contrary, it may harm the solution in terms of giving a higher value for the variance component or the reference variance.

In cases no. 5, 6, and 7 only one benchmark value or control point was used to constrain the least squares solution and the other one was estimated as an unknown parameter. The main purpose of these three tests is to understand the impact of each control point on the global accuracy in terms of the variance component and the parameters accuracy in terms of their standard deviations. In general, these three cases exploit the idea of check points or the general notion of cross-validation. It should be noted that cases no. 5 and 6 use un-weighted or identity weight matrix for the least squares solution.

In case no. 5 BM X (100) was used as a fixed or constant control point without using its stochastic properties and BM Y (107.5) was introduced as an unknown parameter in the least squares solution. In this case the variance component was improved by an order of magnitude (0.0005) in light of the previous cases (1, 2, and 4). Also the standard deviations of the three parameters (A, B, C) were improved. More importantly, BM Y has an accuracy figure of (± 0.021) and an estimated value of (107.414), which is different from its given value (107.5) by 0.086. This difference can be viewed as an external accuracy for this particular point (BM Y). It should be noted that the internal accuracy of BM Y (± 0.021) does not predict its external accuracy (0.086). In general, this case reveals that the use of one control point (BM X) improve the global fit in terms of the variance component as well as the accuracy of A, B, and C.

In case no. 6, BM Y (107.5) was used as a fixed or constant control point in a similar way as we did for BM X in case no. 5. Similar variance component and standard deviations for the parameters were obtained for this case as the ones that were obtained for case no. 5. Moreover, it is very interesting to note that similar external (0.086) and internal (± 0.021) accuracies were obtained for BM X as the ones that were obtained for BM Y in case no. 5. This suggests that there is no reason to prefer one point over the other to serve as a fixed control information to constrain the least squares solution. Either one of them could equally play the role of acceptable control information to constrain the least squares solution.

In case no. 7, the least squares solution was weighted by the distances shown in Table 3. This weighting scheme generates a smaller variance component (0.0002) when compared with the ones that were obtained for cases no. 5 and 6 (0.0005). This decreased in the variance component can be viewed as a positive sign for the use of the distance for least squares weighting. On the other hand, comparable accuracy figures were obtained as the ones that were derived for cases no. 5 and 6.

Cases no. 8, 9, and 10 demonstrate the use of the Gauss-Markov with pseudo observations, which provides a proper handling for the stochastic properties of the new observations and the given ones for control information. The control information (here BM X and BM Y) will be introduced as unknowns in the design matrix (A) and this matrix will become a rank deficient. This rank deficiency was removed by adding two rows or constraints to the design matrix (A). In the least squares solutions these constraints will be integrated with their associated weights. The addition of the pseudo observations can be viewed as prior knowledge with uncertainties or stochastic constraints. In these three cases (8, 9, and 10), three different weighting approaches were tested, namely, un-weighted, weighting by distance, and weighting by variances.

In case no. 8, which is un-weighted solution or equally weighted by an identity matrix, the variance component (0.001) is a little bit higher by an order of a magnitude than the one that was obtained for case no. 7 (0.0002). In light of the previous cases, this is an expected result for the variance component since the un-weighted solutions gave higher variance components when compared with the ones that were weighted by distances. Similar to cases no. 5 and 6, BM X and BM Y have equal standard deviations (0.024) but a little bit larger than the ones in the stated cases (0.021). It is very interesting to observe that the internal accuracy in terms of the standard deviations for BM X and BM Y are very close to their external accuracy since their estimated values (100.030 and 107.470) are differing from their given values (100.00 and 107.50) by very close amounts from the ones that were obtained from the lease squares solution. In other words and with proper modeling of the control information, the least squares internal accuracy can predict its external accuracy.

Case no. 9, which is weighted by distances, shows a dramatic improvement in the variance component by an order of magnitude. It went from (0.001) in case no. 8 to (0.0003) in case no. 9. On the other hand, there is an increase in the standard deviations of the estimated parameters (A, B, C, BM X, and BM Y).

In case no. 10, which is weighted by the variances of the observations and the control information, a very large variance component was obtained. On the other hand, very reasonable results were obtained for the standard deviations of the estimated parameters. Surprisingly enough, in this case the internal accuracy or the standard deviations (± 0.029) from the least squares solution for the two control points (BM X and BM Y) gives a perfect prediction for their external accuracy since they differ by the same amount from their given values.

Cases no. 11, 12, and 13 demonstrate the use of free-network solution for the leveling network. Cases no. 11 and 13 use the minimum-norm or the inner's constraint solution, which is expressed by the D_1 matrix. This matrix has the following structure:

$$D_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Table 5. Estimated parameters for the thirteen cases

Case No.	A	B	C	BM X	BM Y	$\hat{\sigma}_o^2$
1	105.141 ± 0.031	104.483 ± 0.033	106.188 ± 0.031	N/A	N/A	0.0025
2	105.151 ± 0.033	104.489 ± 0.030	106.197 ± 0.030	N/A	N/A	0.0010
3	105.161 ± .033	104.497 ± .026	106.209 ± 0.026	N/A	N/A	146.3056
4	105.150 ± .033	104.489 ± .030	106.197 ± .030	N/A	N/A	0.0019
5	105.092 ± .019	104.421 ± .021	106.138 ± .019	N/A	107.414 ± 0.021	0.0005
6	105.178 ± .017	104.508 ± 0.016	106.225 ± .016	100.086 ± .021	N/A	0.0005
7	105.180 ± 0.018	104.509 ± 0.015	106.227 ± 0.016	100.090 ± 0.024	N/A	0.0002
8	105.137 ± 0.030	104.470 ± 0.031	106.183 ± 0.030	100.030 ± 0.026	107.470 ± .026	0.0010
9	105.137 ± 0.039	104.467 ± .038	106.184 ± 0.038	100.039 ± 0.036	107.461 ± 0.036	0.0003
10	105.147 ± 0.035	104.477 ± 0.032	106.193 ± 0.032	100.029 ± 0.029	107.471 ± 0.029	75.3115
11	0.479 ± 0.012	-0.192 ± 0.012	1.525 ± 0.012	-4.613 ± 0.014	2.801 ± 0.012	0.0005
12	0.000 ± 0.023	-0.670 ± 0.028	1.047 ± 0.029	-5.092 ± 0.029	2.322 ± 0.028	0.0005
13	0.479 ± 0.012	-0.192 ± 0.011	1.526 ± 0.011	-4.611 ± 0.017	2.799 ± 0.012	0.0002

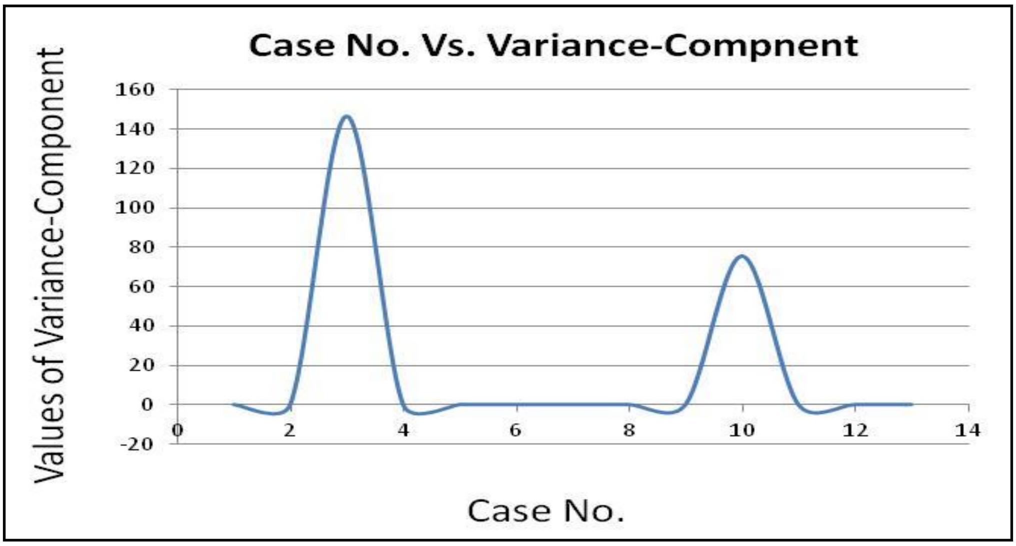

Fig. 3. The variance component for the thirteen cases.

On the other hand, case no 11 is un-weighted and case no. 13 is weighted by distances. Case no. 12 uses a special generalized-inverse, which is expressed by the D_2 matrix and it sets the value of the first unknown to zero. D_2 is a minimum constraint and not an inner's constraint. This matrix has the following structure:

$$D_2 = [1 \quad 0 \quad 0 \quad 0 \quad 0]$$

Although cases no. 11 and 12 use different generalized inverses in terms of the D matrix, it is very interesting to observe that the variance components in both cases are very identical (0.0005), which is one of the theoretical results of the generalized inverse. On the other hand, their standard deviations of the estimated parameters are relatively large when compared with the ones that were obtained from case no. 13.

Case no. 13 has the smallest variance component and standard deviations for the estimated parameters among all cases, which is expected since it uses the minimum norm solution and the distance-based weighting scheme. On the other hand, its variance component is very identical to the one that was obtained for case no. 7 in which the least squares solution is constrained by BM Y as a fixed constraint.

It is very important to note that the solution vector for the network parameters in cases no. 11 and 13 satisfy the constraint shown in Equation 11, which can be interpreted as the minimum-norm solution. In other words, the summation of the estimated parameters is equal to zero.

5. CONCLUSIONS

This paper argues that proper integration and evaluation of existing control information with the adjustment of a new leveling network require a step-wise approach that could reveal the hidden aspects of their uncertainties or stochastic properties. Experimental findings strongly support this argument. The free-network solution delivers the best global accuracy in terms of reference variance or variance component and the best or the smallest standard deviations for the unknown parameters (see case no. 13). This case can be used as a reference or a baseline to compare other cases and results that were obtained during the course of this study as well as practical uses. In particular, the free network solution with minimum-norm could be viewed as the first step in the workflow for the adjustment of the leveling network since it gives a clear idea about the overall accuracy of the new observations as well as the best accuracy for the unknown parameters.

In general, the classical distance-based weighting gives the best global accuracy in terms of the variance component in all cases. Except for cases no. 3 and 10, the ordinarily least squares solution that uses the two control points (BM X and BM Y) does not give the best overall accuracy. The use of a one control point brings a dramatic improve to the estimated variance component (case no. 7), which is very identical to the one that was obtained by the free-network solution (case no. 13). On the other hand, the internal accuracy in this case does not predict its external accuracy. In other words, this modeling does not fully account for the underlying sources of random errors or it has a bias. The least squares solution with pseudo observation reveals that this solution will deliver a very close global accuracy to the free-network solution. Moreover, its internal accuracy from the least squares solution could predict its external accuracy, which was defined by the comparison with the check points.

This work identifies a knowledge gap that could explain and model the connection between Equation 12 and the variance-covariance matrix. In general, this paper calls for more work that could explain the conceptual and the implementation aspects of adjustment computations in practical examples.

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